Introduction to Macro

- Perhaps the most important question that we care about in macro: Distribution of income: across geographic areas, individuals, how it changes over time, income shares, etc.

- Striking observation: a lot of regularities in the past one or two centuries - over long horizons:

  - Kaldor Facts:
    - Labor productivity has grown at a sustained rate in developed economies
    - Capital per worker has also grown
    - Real interest rate has been stable
    - Capital-output ratio has been stable
    - Factor shares have been stable
    - growth rates of labor productivity highly variable across countries
  - Recent arguments and observations by Piketty (2014), or Karabarbounis and Neiman (2014)
  - Convergence and cross-country observations

- Certain regularities over short periods of time:

  - Burns and Mitchell (1946) "facts":
    - $\sigma_{\text{Consumption-nondurable}} \approx \sigma_{\text{Output}}$
    - $\sigma_{\text{Consumption-durable}} > \sigma_{\text{Output}}$
    - $\sigma_{\text{investment}} \approx 3\sigma_{\text{Output}}$
    - $\sigma_{\text{Government purchases}} < \sigma_{\text{output}}$
    - $\sigma_{\text{Total hours workers}} \approx \sigma_{\text{output}}; \sigma_{\text{hours worked per worker}} \ll \sigma_{\text{output}}; \sigma_{\text{Employment}} \approx \sigma_{\text{output}}$
    - $\sigma_{\text{capital}} \ll \sigma_{\text{output}}$

- How can we even think about these issues? Need a theoretical framework.

- Old Macro: specify equations - Lucas Critique

- Modern macro: build from bottom up: specify agents with preferences, technology of production, prices and general equilibrium

- Standard Macro Model:

  - Time: $t = 0, 1, \cdots$
  - Commodities: capital, output (or consumption good), leisure/labor and investment good in each period -infinitely many commodities
  - Firms:
    - producers of consumption goods: $j = 1, \cdots, J_c$, production is given by $y^j_{t,c} = F^c_{j_c,t}(k^j_c, n^j_c)$

    * producers of investment goods: $j = 1, \cdots, J_k$, production is given by $y^j_{t,k} = F^k_{j_k,t}\left(k^j_k, n^j_k\right)$
- Households: $j \in \{0, \cdots, J\}$ preference over consumption, leisure, investment $U^i\left(\mathbf{z}^i\right)$ with $\mathbf{z}^i = \left(\{c^i_t, \ell^i_t, x^i_t, k^i_t\}^{\infty}_{t=0}\right)$, own capital and labor. More formally, U^j is a function from ℓ^p_4 (space of infinite sequences on $\mathbb{R}$ equipped with an appropriate norm say $d\left(x, y\right) = \sum^{\infty}_{n=0} |x_n - y_n|^p$) into $\mathbb{R}$ and is continuous with respect to these sequenes
- Endowments:
    * endowment of leisure: e^i_t
    * time-0 capital: k^i_{-1}
    * shares in firms: $\theta^j_{i,c}, \theta^j_{i,k}$ with

$$\sum_{i=1}^{I} \theta^j_{i,c} = 1, \sum_{i=1}^{I} \theta^j_{i,c} = 1$$

- prices and markets:
    * time-0 trading: trading is done before any consumption or investment takes place; this is sort of weird but you will show in your homework that it will be equivalent to more reasonable market structure
    * market for consumption good at t: price is given by $p_{c,t}$
    * market for capital goods at t: price is given by $p_{x,t}$
    * market for renting out labor at t: price is given by w_t
    * market for renting of capital at t: price is given by r_t
- law of motion for capital:

$$k^i_{t+1} = k^i_t\left(1 - \delta\right) + x^i_t$$

  where x^i_t is the amount of investment goods purchased by household i
- Markets have to clear or feasibility constraints have to hold

$$\sum_{i=1}^{I} c^i_t = \sum_{j=1}^{J_c} y^j_{t,c}$$

$$\sum_{i=1}^{I} x^i_t = \sum_{j=1}^{J_k} y^j_{t,k}$$

$$\sum_{i=1}^{I} k^i_t = \sum_{j=1}^{J_c} k^j_{t,c} + \sum_{j=1}^{J_k} k^j_{t,k}$$

$$\sum_{i=1}^{I} e^i_t - \ell^i_t = \sum_{j=1}^{J} n^j_{t,c} + \sum_{j=1}^{J_k} n^j_{t,k}$$

- Definition of competitive equilibrium: sequence of allocations $\hat{\mathbf{z}}^i = \left\{\hat{c}^i_t, \hat{\ell}^i_t, \hat{k}^i_t, \hat{x}^j_t\right\}_{t=0,\cdots,\infty, j=1,\cdots,J}$, $\left\{\hat{y}^j_{t,c}, \hat{k}^j_{t,c}, \hat{n}^j_{t,c}\right\}_{j=1,\cdots,J_c, t=0,\cdots,\infty}$, $\left\{\hat{y}^j_{t,k}, \hat{k}^j_{t,k}, \hat{n}^j_{t,k}\right\}_{j=1,\cdots,J_k, t=0,\cdots,\infty}$ and prices $\{\hat{p}_{c,t}, \hat{p}_{x,t}, \hat{r}_t, \hat{w}_t\}^{\infty}_{t=0}$ constitutes a competitive equilibrium such that:

* Households maximize taking prices and firms' profits as given. That is they solve the following optimization

$$\hat{\mathbf{z}}^i \in \arg\max_{z^i} U^i\left(\mathbf{z}^i\right)$$

subject to

$$\sum_{t=0}^{\infty}\left[\hat{p}_{c,t}c_t^i + \hat{p}_{x,t}x_t^i\right] \leq \sum_{t=0}^{\infty}\left[\hat{w}_t\left(e_t^i - \ell_t^i\right) + \hat{r}_t k_t^i\right] + \Pi^i$$
$$k_{t+1}^i = k_t^i(1-\delta) + x_t^i$$
$$c_t^i, x_t^i, \ell_t^i, k_t^i \geq 0$$
$$k_0^i \text{ :given}$$

where Π^i is total profits from firms.

* Firms maximize taking prices as given. They solve the following optimization problems
  · Consumption good producers

$$\left\{\hat{k}_t^j, \hat{n}_t^j\right\} \in \arg\max_{k,n} \hat{p}_{c,t}F_{t,c}^j(k,n) - \hat{w}_t n - \hat{r}_t k$$

  · Investment good producers

$$\left\{\hat{k}_t^j, \hat{n}_t^j\right\} \in \arg\max_{k,n} \hat{p}_{x,t}F_{t,k}^j(k,n) - \hat{w}_t n - \hat{r}_t k$$

* Allocations are feasible
* Profits are consistent with firm behavior

$$\Pi^j = \sum_{t=0}^{\infty}\sum_{j=1}^{J_c}\theta_{i,c}^j\left[\hat{p}_{c,t}\hat{y}_{t,c}^j - \hat{w}_t\hat{n}_{t,c}^j - \hat{r}_t\hat{k}_{t,c}^j\right]$$
$$+ \sum_{t=0}^{\infty}\sum_{i_k=1}^{J_k}\theta_{i,k}^j\left[\hat{p}_{x,t}\hat{y}_t^j - \hat{w}_t\hat{n}_{t,k}^j - \hat{r}_t\hat{k}_{t,k}^j\right]$$

– It is worth noting that arrangement of ownership and production in this economy is not exactly the same thing that we see in the real world. In reality households own shares in firms and then firms decide about purchasing structures and equipment and investment goods. Here, capital is directly owned by households and is rented out to firms to use for production. It turns out, if we switch to that alternative view of the world, nothing changes. The equilibrium definition is a bit more involved but allocations do not change. This is because of the so-called complete market assumption. That is, the market structure that we have imposed is quite general even though it is weird. Any allocation that is achievable with some other trading mechanism can be also achieved in our time-0 trading mechanism.

- Notation: for simplicity we refer to an allocation with $\mathbf{z}$ which includes consumption, leisure, investment, capital of all households and firm's allocation. We will refer to the set of feasible allocations as $\mathcal{Z}$.
- What is not in this?
  - Finance: no one is borrowing or lending (see homework for this); no one is trading shares of firms
  - Wiggles: will introduce later
  - Government: will introduce later
  - Externalities: will introduce later
  - Rest of the world: probably wont have time to talk about - maybe in the homeworks!
- First Welfare theorem
  - Computing CE is hard; have to solve for infinite sequences of prices and allocations
  - We can use the first welfare theorem to make it easier to solve

Theorem 1. *Suppose that a sequence of allocations* $\left\{\hat{c}_t^j, \hat{\ell}_t^j, \hat{k}_t^j, \hat{x}_t^j\right\}_{t=0,\cdots,\infty, j=1,\cdots,J}$, $\left\{\hat{y}_t^{i_c}, \hat{k}_t^{i_c}, \hat{n}_t^{i_c}\right\}_{i_c=1,\cdots,I_c, t=0,\cdots,\infty}$, $\left\{\hat{y}_t^{i_k}, \hat{k}_t^{i_k}, \hat{n}_t^{i_k}\right\}_{i_k=1,\cdots,I_k, t=0,\cdots,\infty}$ *and prices* $\{\hat{p}_{c,t}, \hat{p}_{x,t}, \hat{r}_t, \hat{w}_t\}_{t=0}^{\infty}$ *constitutes a CE. Suppose further that* $U^j(\cdot)$ *is locally non-satiated*[1] *and that*

$$\sum_{t=0}^{\infty} \hat{p}_{c,t}\hat{c}_t^j + \hat{p}_{x,t}\hat{x}_t^j + \hat{w}_t\ell_t^j < \infty, \forall j = 1, \cdots, J.$$

Then the allocation, $\left\{\hat{c}_t^j, \hat{\ell}_t^j, \hat{k}_t^j, \hat{x}_t^j\right\}_{t=0,\cdots,\infty, j=1,\cdots,J}$, $\left\{\hat{y}_t^{i_c}, \hat{k}_t^{i_c}, \hat{n}_t^{i_c}\right\}_{i_c=1,\cdots,I_c, t=0,\cdots,\infty}$, $\left\{\hat{y}_t^{i_k}, \hat{k}_t^{i_k}, \hat{n}_t^{i_k}\right\}_{i_k=1,\cdots,I_k, t=0,\cdots,\infty}$ *is Pareto optimal.*

Proof. Suppose not. That is, suppose there exists an alternative allocation $\left\{\tilde{c}_t^j, \tilde{\ell}_t^j, \tilde{k}_t^j, \tilde{x}_t^j\right\}_{t=0,\cdots,\infty, j=1,\cdots,J}$, $\left\{\tilde{y}_t^{i_c}, \tilde{k}_t^{i_c}, \tilde{n}_t^{i_c}\right\}_{i_c=1,\cdots,I_c, t=0,\cdots,\infty}$, $\left\{\tilde{y}_t^{i_k}, \tilde{k}_t^{i_k}, \tilde{n}_t^{i_k}\right\}_{i_k=1,\cdots,I_k, t=0,\cdots,\infty}$ which is feasible and satisfies

$$U^j\left(\left\{\tilde{c}_t^j, \tilde{\ell}_t^j\right\}\right) \geq U^j\left(\left\{\hat{c}_t^j, \hat{\ell}_t^j\right\}\right), \forall j = 1, \cdots, J$$

with at least one inequality being strict. Without loss of generality assume that this is household number $j = 1$. Then by definition of competitive equilibrium it must be that

$$\sum_{t=0}^{\infty}\left[\hat{p}_{c,t}\tilde{c}_t^j + \hat{p}_{x,t}\tilde{x}_t^j + \hat{w}\tilde{\ell}_t^j - \hat{r}_t\tilde{k}_t^j\right] \geq \sum_{t=0}^{\infty}\hat{w}_t e_t^j + \hat{\Pi}^j$$

$$\sum_{t=0}^{\infty}\left[\hat{p}_{c,t}\tilde{c}_t^1 + \hat{p}_{x,t}\tilde{x}_t^1 + \hat{w}\tilde{\ell}_t^1 - \hat{r}_t\tilde{k}_t^1\right] > \sum_{t=0}^{\infty}\hat{w}_t e_t^1 + \hat{\Pi}^1$$

[1]Formally a function $f(x)$ that maps any space X (take it to be a Hilbert space) into $\mathbb{R}$ is locally non-satiated if for any $x \in X$ and $\varepsilon > 0$, there is $x' \in X$ such that $d(x, x') < \varepsilon$ and $f(x') > f(x)$. Informally, it means that the individual does not get satiated at some point.

Question: Why?

Note that also that by optimality of firms' decisions it must be that

$$\hat{\Pi}^j \geq \tilde{\Pi}^j$$

where $\tilde{\Pi}^j$ is the total profits accrued to household j using prices in the competitive equilibrium (Why?). We therefore have

$$\sum_{t=0}^{\infty} \left[\hat{p}_{c,t}\tilde{c}_t^j + \hat{p}_{x,t}\tilde{x}_t^j + \hat{w}\tilde{\ell}_t^j - \hat{r}_t\tilde{k}_t^j\right] \geq \sum_{t=0}^{\infty} \hat{w}_t e_t^j + \tilde{\Pi}^j$$

$$\sum_{t=0}^{\infty} \left[\hat{p}_{c,t}\tilde{c}_t^1 + \hat{p}_{x,t}\tilde{x}_t^1 + \hat{w}\tilde{\ell}_t^1 - \hat{r}_t\tilde{k}_t^1\right] > \sum_{t=0}^{\infty} \hat{w}_t e_t^1 + \tilde{\Pi}^1$$

If we sum over the above, we get

$$\sum_{j=1}^{J}\sum_{t=0}^{\infty} \left[\hat{p}_{c,t}\tilde{c}_t^j + \hat{p}_{x,t}\tilde{x}_t^j + \hat{w}\tilde{\ell}_t^j - \hat{r}_t\tilde{k}_t^j\right] > \sum_{j=1}^{J}\sum_{t=0}^{\infty} \hat{w}_t e_t^j + \tilde{\Pi}^j \tag{1}$$

In a homework, you are asked to show that

$$\begin{aligned}\sum_{j=1}^{J} \tilde{\Pi}^j =& \sum_{t=0}^{\infty}\sum_{i_c=1}^{I_c} \hat{p}_{c,t}\tilde{y}_t^{i_c} - \hat{w}_t\tilde{n}_t^{i_c} - \hat{r}_t\tilde{k}_t^{i_c} \\ &+ \sum_{t=0}^{\infty}\sum_{i_c=1}^{I_c} \hat{p}_{x,t}\tilde{y}_t^{i_k} - \hat{w}_t\tilde{n}_t^{i_k} - \hat{r}_t\tilde{k}_t^{i_k}\end{aligned} \tag{2}$$

Now these are a bunch of infinite sums and without knowing something about them we cannot easily rearrange them. However, because they are all finite, we can do the rearranging and write

$$\begin{aligned}\sum_{j=1}^{J}\sum_{t=0}^{\infty} \hat{w}_t e_t^j + \tilde{\Pi}^j =& \sum_{t=0}^{\infty}\Bigg[\hat{p}_{c,t}\sum_{i_c=1}^{I_c}\tilde{y_t}^{i_c} + \hat{p}_{x,t}\sum_{i_{k=1}}^{I_k}\tilde{y}_t^{i_k} \\ &-\hat{r}_t\left(\sum_{i_c=1}^{I_c}\tilde{k}_t^{i_c} + \sum_{i_k=1}^{I_k}\tilde{k}_t^{i_k}\right) - \hat{w}_t\left(\sum_{i_c=1}^{I_c}\tilde{n}_t^{i_c} + \sum_{i_k=1}^{I_k}\tilde{n}_t^{i_k} - \sum_{j=1}^{J} e_t^j\right)\Bigg]\end{aligned} \tag{3}$$

Using a similar logic, we can write the left hand side of (1):

$$\sum_{t=0}^{\infty}\left[\hat{p}_{c,t}\sum_{j=1}^{J}\tilde{c}_t^j + \hat{p}_{x,t}\sum_{j=1}^{J}\tilde{x}_t^j + \hat{w}_t\sum_{j=1}^{J}\tilde{\ell}_t^j - \hat{r}_t\sum_{j=1}^{J}\tilde{k}_t^j\right] \tag{4}$$

We can use feasibility constraint as defined above and write (3) as

$$\sum_{j=1}^{J}\sum_{t=0}^{\infty} \hat{w}_t e_t^j + \tilde{\Pi}^j = \sum_{t=0}^{\infty}\left[\hat{p}_{c,t}\sum_{j=1}^{J}\tilde{c}_t^j + \hat{p}_{x,t}\sum_{j=1}^{J}\tilde{x}_t^j - \hat{r}_t\sum_{j=1}^{J}\tilde{k}_t^j + \hat{w}_t\sum_{j=1}^{J}\tilde{\ell}_t^j\right]$$

This is exactly the same as the left hand side (4). Thus we have an inequality that states the above expression must be strictly less than itself. This is a contradiction. □

- The idea behind the first welfare theorem is what we mentioned in class. If there is something that makes everyone better off. Then they must not be able to afford it and if no one can afford an allocation it must not be feasible.

- The assumption that market value of allocations must be finite is a binding assumption. In fact, in overlapping generations model which I believe you will learn from Steve Spear, it is not hard to come up with examples where it breaks down and the CE is not pareto optimal.

- The first welfare theorem has two main implications:

  - Substantive implication: First, letting markets do their work create an optimal allocation. It would be impossible to improve everyone's life! Of course it is possible that society as a whole likes other things. For example, if there are some people whose endowment of capital and time is very low, they are miserable in a CE. Now if society likes to provide a sufficient level of utility to these guys, they must deviate from CE. FWT is useful because it identifies these trade-offs. it would mean that if we want to keep these households happy, someone must pay for it - one job for the government would be to enforce this but markets can also provide this service through charities perhaps.
  - Second practical implication: In solving for allocations, we wont have to worry about prices and just focus on allocations. This makes life easy in that it allows us to have an easier job of solving for the fixed point problem of guessing certain prices and then find prices so that markets clear. Instead we focus on solving a Pareto problem as I describe below.

- Pareto problem: it can be shown that if preferences are strictly concave then finding a pareto optimal allocation is equivalent to solving the following problem

$$\max_{z\in\mathcal{Z}}\sum_{j=1}^{J}\alpha^j U^j\left(\{c_t^j,\ell_t^j\}\right)$$

  where $\alpha^j > 0$ is the welfare weight of household j. It is kind of saying that the allocations in a CE are equivalent to those in a planned economy where a fictitious planner just tells people how much to consume and invest and work and the planner cares about them at rate α^j.

- To be more precise, an allocation $\hat{z}\in\mathcal{Z}$ is pareto optimal if and only if there exists positive welfare weights $\alpha^j \geq 0$ such that

$$\hat{z}\in\arg\max_{z\in\mathcal{Z}}\sum_{j=1}^{J}\alpha^j U^j\left(\{c_t^j,\ell_t^j\}\right)$$

Proof. The proof that the solution of the above problem is pareto optimal is straightforward. To see this, consider a $\hat{z}$ that solves the above maximization problem and suppose that an alternative allocation,$\tilde{z}$, exists which makes everyone weakly better off and some types strictly better off. If

$\alpha^j > 0$ for all j, then obviously this cannot be since the objective in the above optimization is higher under $\tilde{z}$ than z. When some of the α^j's are zero, then it must be that $\tilde{z}$ and $\hat{z}$ give the same level of utility. Since U^j is strictly concave, a convex combination of z and $\tilde{z}$ would deliver a higher utility. Therefore, this is a contradiction.

Now suppose $\tilde{z}$ is a pareto optimal allocation. Define the following set

$$A = \left\{ \boldsymbol{u} = \left(u^1, \cdots, u^J\right) \middle| \exists z \in \mathcal{Z}, u^j = U^j(z) \right\}$$

where we have abused the notation a bit by defining $U^j(z)$ to be U^j evaluated at the sequence of consumption and leisure for household j. Since the utility function is concave, then the above set must be a convex set (why?). Define set B as follows

$$B = \left\{ \boldsymbol{u} = \left(u^1, \cdots, u^J\right) \middle| \forall j, u^j \geq U^j(\tilde{z}) \right\}$$

Since $\tilde{z}$ is pareto optimal, it must be that

$$A \cap B = \left\{ \left(U^1(\tilde{z}), U^2(\tilde{z}), \cdots, U^J(\tilde{z}) \right) \right\}$$

(why?). Since A and B are both convex sets in $\mathbb{R}^J$, we can apply the separating hyperplane theorem. As a result, vectors $\boldsymbol{\alpha} \in \mathbb{R}^J, \boldsymbol{\alpha} \neq 0$ and a constant c must exist such that

$$\langle \boldsymbol{\alpha}, \boldsymbol{u} \rangle \geq c \text{ if } \boldsymbol{u} \in B$$
$$\langle \boldsymbol{\alpha}, \boldsymbol{u} \rangle \leq c \text{ if } \boldsymbol{u} \in A$$

where $\langle \cdot, \cdot \rangle$ is the product in $\mathbb{R}^J$:

$$\langle \boldsymbol{\alpha}, \boldsymbol{u} \rangle = \sum_{j=1}^{J} \alpha^j u^j$$

Note that since $\tilde{\boldsymbol{u}} = \left(U^1(\tilde{z}), U^2(\tilde{z}), \cdots, U^J(\tilde{z}) \right)$ belongs to both sets, we must have

$$c = \langle \boldsymbol{\alpha}, \tilde{\boldsymbol{u}} \rangle$$

This together with the statement of the separating hyperplane theorem implies that

$$\langle \boldsymbol{\alpha}, \boldsymbol{u} \rangle \leq \langle \boldsymbol{\alpha}, \tilde{\boldsymbol{u}} \rangle, \forall \boldsymbol{u} \in A$$

and by definition of A

$$\sum_j \alpha^j U^j(z) \leq \sum_j \alpha^j U^j(\tilde{z}), \forall z \in \mathcal{Z}$$

which is equivalent to

$$\tilde{z} \in \arg\max_{z \in \mathcal{Z}} \sum \alpha^j U^j(z)$$

The only thing that remains to be shown is that $\alpha^j \geq 0$. To show this, let $i \in \{1, \cdots, J\}$ and

$$\boldsymbol{e}^i = \left(\underbrace{0, \cdots, 0}_{i-1 \text{ times}}, 1, 0, \cdots, 0 \right)$$

Then since $\tilde{\boldsymbol{u}} + \boldsymbol{e}^i \in B$, we must have that

$$c = \langle \boldsymbol{\alpha}, \tilde{\boldsymbol{u}} \rangle \leq \left\langle \boldsymbol{\alpha}, \tilde{\boldsymbol{u}} + \boldsymbol{e}^i \right\rangle = \langle \boldsymbol{\alpha}, \tilde{\boldsymbol{u}} \rangle + \alpha^i \Rightarrow 0 \leq \alpha^i$$

which completes the proof. □

- **Aggregation of households:** Note that when utilities are time-separable and people have the same discount rate, the objective in the above optimization can be written as

$$\sum_{t=0}^{\infty} \beta^t \sum_{j=1}^{J} \alpha^j u^j \left(c_t^j, \ell_t^j\right)$$

This is basically saying that the economy we described is equivalent to an economy where a single representative household is making decisions about how much to consume, work and invest. This even further simplifies our life in solving this problem. Note that it does not necessarily imply that distributions do not matter. In particular, we can define the following aggregate utility function

$$U\left(c, \ell; \alpha\right) = \max \sum_{j=1}^{J} \alpha^j u^j \left(c^j, \ell^j\right)$$

subject to

$$\sum_{j=1}^{J} c^j = c, \sum_{j=1}^{J} \ell^j = \ell$$

Then the planning problem is simply given by

$$\max_{c_t, \ell_t, k_t^{i_c}, n_t^{i_c}, k_t^{i_k}, n_t^{i_k}} \sum_{t=0}^{\infty} \beta^t U\left(c_t, \ell_t; \alpha\right) \tag{P}$$

subject to

$$\begin{aligned} c_t &= \sum_{i_c=1}^{I_c} F^{i_c}\left(k_t^{i_c}, n_t^{i_c}\right) \\ x_t &= \sum_{i_k=1}^{I_k} F^{i_k}\left(k_t^{i_k}, n_t^{i_k}\right) \\ k_{t+1} &= (1-\delta) k_t + x_t \\ \bar{e} - \ell &= \sum_{i_c=1}^{I_c} n_t^{i_c} + \sum_{i_k=1}^{I_k} n_t^{i_k} \\ k_t &= \sum_{i_c=1}^{I_c} k_t^{i_c} + \sum_{i_k=1}^{I_k} k_t^{i_k} \end{aligned}$$

As you can seen this problem at least gets rid of a whole bunch of heterogeneity inherent in our initial problem. Now there is a class of utility functions that makes life even simpler. Suppose, for example that

$$u^j\left(c, \ell\right) = \frac{\left(c^\theta \ell^{1-\theta}\right)^{1-\sigma}}{1-\sigma}, \forall j$$

Then we have

$$U(c,\ell;\alpha)=\max\sum_{j=1}^{J}\alpha^{j}\frac{\left(c_j^{\theta}\ell_j^{1-\theta}\right)^{1-\sigma}}{1-\sigma}$$

subject to

$$\sum_{j=1}^{J}c_j=c,\sum_{j=1}^{J}\ell_j=\ell$$

Taking first order conditions from this problem we have

$$\alpha^{j}\theta\frac{\left(c_j^{\theta}\ell_j^{1-\theta}\right)^{1-\sigma}}{c_j}=\lambda_c,\alpha^{j}\left(1-\theta\right)\frac{\left(c_j^{\theta}\ell_j^{1-\theta}\right)^{1-\sigma}}{\ell_j}=\lambda_l$$

Let's do some algebra now. If we divide the two equations, we get

$$\frac{\ell_j}{c_j}=\frac{1-\theta}{\theta}\frac{\lambda_c}{\lambda_l},\forall j=1,\cdots,J$$

This means that

$$\frac{\ell_j}{c_j}=\frac{\ell}{c}=\frac{1-\theta}{\theta}\frac{\lambda_c}{\lambda_l}$$

Now if define this ratio to be some constant κ, then we have

$$\alpha^{j}\theta c_j^{-\sigma}\kappa^{1-\sigma}=\lambda_c\to\frac{c_j}{c_i}=\left(\frac{\alpha^{j}}{\alpha^{i}}\right)^{\frac{1}{\sigma}}\to c_i=\frac{\left(\alpha^{i}\right)^{\frac{1}{\sigma}}}{\sum_{j=1}^{J}\left(\alpha^{j}\right)^{\frac{1}{\sigma}}}c$$

Similarly

$$\ell_i=\frac{\left(\alpha^{i}\right)^{\frac{1}{\sigma}}}{\sum_{j=1}^{J}\left(\alpha^{j}\right)^{\frac{1}{\sigma}}}\ell$$

Thus the utility is given by

$$\begin{aligned}U(c,\ell;\alpha)&=\sum_{j=1}^{J}\alpha^{j}\frac{\left(\alpha^{j}\right)^{\frac{1-\sigma}{\sigma}}\left(c^{\theta}\ell^{1-\theta}\right)^{1-\sigma}}{\left(1-\sigma\right)\left(\sum_{j=1}^{J}\left(\alpha^{j}\right)^{\frac{1}{\sigma}}\right)^{1-\sigma}}\\&=\left[\sum_{j=1}^{J}\left(\alpha^{j}\right)^{\frac{1}{\sigma}}\right]^{\sigma}\frac{\left(c^{\theta}\ell^{1-\theta}\right)^{1-\sigma}}{1-\sigma}\end{aligned}$$

This implies that we can write the objective function in the planning problem P, as

$$\left[\sum_{j=1}^{J}\left(\alpha^{j}\right)^{\frac{1}{\sigma}}\right]^{\sigma}\sum_{t=0}^{\infty}\beta^{t}\frac{\left(c_t^{\theta}\ell_t^{1-\theta}\right)^{1-\sigma}}{1-\sigma}$$

This means that distribution of allocations actually do not affect aggregate outcomes since α's have no effect on the optimal decision. So we can just separate the problem of determining optimal allocations from that of their distribution among households.

- This class of utility function is a special case of a more general class of utility functions that satisfy the so-called Gorman aggregation. One can show that this is the case if and only if preferences are homothetic and identical, i.e.,

$$(c, \ell) \sim_j (c', \ell') \Leftrightarrow (\lambda c, \lambda \ell) \sim_j (\lambda c', \lambda \ell'), \forall \lambda > 0$$

- Note that in general the problem is still quite tractable because conditional on the vector α, we can characterize - as we will see - the evolution of aggregate allocations. However, in case of general preferences that are non-homothetic, the distribution of resources interacts with aggregate behavior of the economy. See the paper by Chatterjee (1994).
- A simplification that we have made in our time-separable preferences above was the assumption of constant discounting. This assumption allows us to write the problem recursively in an easy fashion - as we will show. It, however, implies that the rate at which people discount the future is the same independent of where they stand in time. In other word, at $t = 0$, the value a unit of utility at time 6 relative to time 5 is β. Now if the individual goes to period $t = 1$, the same is true. Now suppose instead utilities evaluated at $t = 0$ are given by

$$\sum_{t=0}^{\infty} \beta_t u_t$$

where β_t is not an exponential series. Then we have to specify preferences for an individual who is making decisions at $t = 1$. The bottomline is we cannot come up with a utility function where the two individuals agree on the way they evaluate future utilities. This creates what is known as a time-inconsistency problem in decision making where planning made by an individual at $t = 0$ is not consistent with optimal decisions of that same individual at $t = 1$. You can read more about this in the seminal paper by Laibson (1997).

- **Second Welfare theorem**: While we have talked about pareto optimality of CE, we have not really explained how should one go back from a general P.O. allocation to a C.E. (basically how to construct prices). It turns out to do this generally, we would need transfers. In particular, for a sequence of transfers $\{T^j\}_{j=1}^J$ where $\sum_{j=1}^J T^j = 0$, we modify the budget constraint in the definition of C.E. to

$$\sum_{t=0}^{\infty} \left(p_{c,t} c_t^j + p_{x,t} x_t^j\right) \leq \sum_{j=0}^{\infty} \left[w_t \left(e_t^j - \ell_t^j\right) + r_t k_t^j\right] + \Pi^j + T^j$$

A *C.E. with transfers* given transfers T^j can be defined accordingly using the above modified budget constraint. We thus have the following result, also known as the *Second Welfare Theorem*:

Theorem 2. *Consider a P.O. allocation, $\hat{z} \in \mathcal{Z}$. Then there exists transfers $\left\{\hat{T}^j\right\}_{j=1}^J$ together with a vector of prices $\{\hat{p}_{c,t}, \hat{p}_{x,t}, \hat{w}_t, \hat{r}_t\}$ so that they together with $\hat{z}$ constitute a C.E. with transfers.*

Proof. In order to prove existence of prices, we would need to resort to a separating hyperplane theorem for infinite dimensional spaces which is really not that economically insightful and beyond our scope here - but mathematical very interesting and challenging. You can read the proof of the finite dimensional case in standard microeconomics textbooks such as the one by Mas-colell, Whinston and Green. More general proofs can be found in various papers by Zame, Mas-colell and Bewley. □

- This theorem basically states that we can achieve any pareto allocation we want as long as we have individual specific transfers. Unfortunately, this is too strong of an assumption. In other words, a government or planner, needs to know various details about households' wealth and productivity and other sources of endowment. This is perhaps too costly for the government - almost impossible to verify someone's productivity in the labor market even if we can observe out their wealth. Once it is hard for the government to observe people's characteristics, the second welfare theorem disappears and we do not get to implement any pareto optimal allocation that we want. In fact, this potentially creates a trade-off between the desired allocation of goods and services across people and economic efficiency. This is one of the main starting points for the public finance literature that studies redistributive taxation and its interactions with efficiency. We will come back to this later during the course.

- **Aggregation of production functions:** Standard results in micro allows us to always aggregate production function, again you can read about this in micro textbooks. This implies that we can basically represent the production side of the economy with two production functions:
$$F^c\left(k^c, n^c\right), F^k\left(k^k, n^k\right)$$

- Another assumption that makes our life really simple is to assume that the two production functions above are identical and are both constant returns to scale and are strictly concave. In this case, economic profits Π^j will always be zero in equilibrium and prices of consumption and investment will be equal. We can thus aggregate the above two production functions into just one production function $F(k, n)$ and write feasibility as

$$c_t + x_t = y_t = F\left(k_t, n_t\right)$$

  This is the familiar NIPA equation from intermediate macro - since this is a closed economy without a government, NX and G do not show up here!

- *Question: what is the NIPA equation look like in case of the disaggregated economy?*

- Now that we have significantly simplified this problem, we are ready to try to solve to do this let us restate the problem again

$$\max \sum_{t=0} \beta^t U\left(c_t, \ell_t\right)$$

subject to

$$
\begin{aligned}
c_t + x_t &= F\left(k_t, n_t\right) \\
k_{t+1} &= \left(1-\delta\right) k_t + x_t \\
\overline{e} - n_t &= \ell_t \\
k_0&\text{:given} \\
x_t, c_t, k_t, n_t, \ell_t &\geq 0
\end{aligned}
$$

where we have imposed that aggregate leisure in the economy is constant. One thing that is important to note is that the above programming problem only depends on k_0 - the initial state of the economy. As we will see this makes the analysis tractable - dealing with one state variable is really easy!

- In order to use tools from Dynamic Programming developed by Richard Bellman and presented also in Stokey and Lucas Jr (1989), we need one more step. We define the following auxiliary utility function

  $$\hat{U}\left(k, k'\right) = \max_{c,\ell,n\geq 0} U\left(c, \ell\right)$$

  subject to

  $$
  \begin{aligned}
  c &= F\left(k, n\right) + \left(1-\delta\right) k - k' \\
  n &= \overline{e} - \ell
  \end{aligned}
  $$

  Note that if we assume that the solution of the above problem is interior, then taking first order conditions and combining them, we will have

  $$\frac{U_\ell\left(c, \ell\right)}{U_c\left(c, \ell\right)} = F_n\left(k, \overline{e} - \ell\right)$$

  This equation is quite intuitive. The right hand side is the marginal benefit of one more hours of work. The left hand side is its marginal cost. U_ℓ is the cost of giving up one unit of leisure and by dividing it by U_c we change the units of this cost from being in utility terms to being in consumption terms. We will also refer to the solution of this programming problem as $\tilde{n}\left(k, k'\right), \tilde{c}\left(k, k'\right)$. A special case of this is one where there is labor supply is inelastic, i.e., households dont care about leisure, $u^j_\ell\left(c, \ell\right) = 0$ in which case $\ell = 0$ or $n = \overline{e}$. In this case, $\hat{U}\left(k, k'\right) = U\left(F\left(k, \overline{e}\right) + \left(1-\delta\right) k - k', 0\right) = u\left(F\left(k, \overline{e}\right) + \left(1-\delta\right) k - k'\right)$ - with a slight abuse of notation. For now, let's focus on this case.

- Given this auxiliary utility function, the above programming problem is equivalent to

  $$\max_{k_t} \sum_{t=0}^{\infty} \beta^t u\left(F\left(k_t, \overline{e}\right) + \left(1-\delta\right) k_t - k_{t+1}\right)$$

  subject to

  $$
  \begin{aligned}
  \left(1-\delta\right) k_t \leq k_{t+1} &\leq F\left(k_t, \overline{e}\right) + \left(1-\delta\right) k_t \\
  k_0 &: \text{ given}
  \end{aligned}
  $$

  Question: where are the inequalities coming from?

- **Dynamic Programming:** If we refer to the value of the above optimization problem as $V(k_0)$, then this value function must solve the following functional equation

$$V(k) = \max_{k' \in \Gamma(k)} u\left(F\left(k, \bar{e}\right) + (1-\delta) k - k'\right) + \beta V(k')$$

where $\Gamma(k) = [(1-\delta) k, (1-\delta) k + F(k, \bar{e})]$.

- Standard results - by the end of the semester anyways! - show that
  - if U and F are continuous, then $V(\cdot)$ is unique. The idea is that if we define the following transformation on the space of continuous and bounded functions

$$Tv(k) = \max_{k' \in \Gamma(k)} u\left(F\left(k, \bar{e}\right) + (1-\delta) k - k'\right) + \beta v(k')$$

    for an arbitrary function $v(\cdot)$, then V must be a fixed point of this transformation. The proof basically goes by showing that T is a contraction mapping and therefore has a unique fixed-point which would be V.
  - If U is strictly concave and strictly increasing, F is strictly concave and strictly increasing, then V is strictly concave, and strictly increasing.
  - If U and F are continuously differentiable, then V is also continuously differentiable.
  - For any continuous function, $V_0(k)$, we have

$$\lim_{n \to \infty} T^n V_0(k) = V(k)$$

    where $T^n V_0(k)$ is the result of applying T to V_0, n-times. Also, the convergence occurs according to the sup-norm on the space of functions. This is very useful result for computations.

- If the solution for k' in the optimization problem above is interior, we can write

$$-u'\left(F\left(k, \bar{e}\right) + (1-\delta) k - k'\right) + \beta V'(k') = 0 \tag{5}$$

We refer to the solution of the optimization by $\tilde{k}(k)$, what is also referred to as a *policy function*. Note that we can calculate the derivative of the value function, $V(k)$ by applying the Envelope theorem to the maximization above which implies that

$$V'(k) = u'\left(F\left(k, \bar{e}\right) + (1-\delta) k - \tilde{k}(k)\right) \cdot \left(F_k\left(k, \bar{e}\right) + 1 - \delta\right)$$

We thus have

$$V'\left(\tilde{k}(k)\right) = u'\left(F\left(\tilde{k}(k), \bar{e}\right) + (1-\delta)\tilde{k}(k) - \tilde{k}\left(\tilde{k}(k)\right)\right) \cdot \left(F_k\left(\tilde{k}(k), \bar{e}\right) + 1 - \delta\right)$$

If we replace this in (5) we get

$$u'\left(F\left(k, \bar{e}\right) + (1-\delta) k - \tilde{k}(k)\right) = \beta u'\left(F\left(\tilde{k}(k), \bar{e}\right) + (1-\delta)\tilde{k}(k) - \tilde{k}\left(\tilde{k}(k)\right)\right) \cdot \left(F_k\left(\tilde{k}(k), \bar{e}\right) + 1 -$$

We can rewrite this as

$$u'(c_t) = \beta u'(c_{t+1})\left[1 - \delta + F_k(k_{t+1}, \overline{e})\right]$$

This equation is called an Euler equation and intuitively captures the trade-offs in optimal investment decision. When the consumer at date t gives up ε units of consumption (where ε is small) and invests it, capital stock in period $t+1$ goes up by ε. If we assume that k_{t+2} remains, i.e., period $t+2$ onwards, remains unchanged, then the resources available to the consumer in period $t+1$ are given by $(F_k(k_{t+1}, \overline{e}) + 1 - \delta)\varepsilon$. This is because the extra output produced with the extra units of capital plus the undepreciated value of capital are all available for consumption. The marginal cost (in terms of utils) of cutting consumption at t by ε is $\beta^t u'(c_t)\varepsilon$ while the marginal benefit of increasing consumption at $t+1$ is $\beta^{t+1}u'(c_{t+1}) \times (1 - \delta + F_k(k_{t+1}, \overline{e}))\varepsilon$. Now at the optimum and for a small value for ε, marginal benefit must be equal to the marginal cost and thus we get the Euler equation.

- **Evolution of capital over time:** In order to understand the evolution of capital over time, we need basic characterization of the policy function $\tilde{k}(k)$. We can use the recursive formulation to show that the policy function, $\tilde{k}(k)$, is increasing in k. We can further show that there are two value of k that satisfy $\tilde{k}(k) = k$. One of them is 0 and the other one is some positive value $k^* = \tilde{k}(k^*)$. It can then be shown that if $k_0 > 0$, then

$$\lim_{n\to\infty} \tilde{k}^n(k_0) = k^*$$

Now this result is pretty important, because it states that capital accumulation on its own cannot generate long-term growth. In other words, the fact that we have seen persistent growth in the past 200 years through the lens of this model can only be explained if we were on a very long and slow transition path to the steady states. There are various reasons to believe that this cannot be true. We will next discuss various implications of this growth model.

  - In words, the only we can have growth in this model is by accumulation of capital. For high enough values of capital, decreasing returns to scale kick in and do not let the economy grow
  - To summarize, this model - in Steady State - explains some of the Kaldor facts, namely that factor shares are stable and that capital labor is stable but it cannot generate growth in GDP per capita - aside from transition - and in real wages. To calculate factor prices which with a little abuse of notation we call w_t and r_t , we must solve the firm's problem

$$\max_{k,n} F(k, n) - w_t n - r_t k$$

We therefore have

$$F_n(k_t, \overline{e}) = w_t$$
$$F_k(k_t, \overline{e}) = r_t$$

where $k_t = \tilde{k}^t(k_0)$. This implies that in the steady state, real wages and real interest rates are constant - the second one is a Kaldor facts. Over the course of transition, real wages rise and real interest rates fall. Moreover, factor shares are given by

$$\alpha_{k,t} = \frac{r_t k_t}{F(k_t, \bar{e})} = \frac{F_k(k_t, \bar{e})\, k_t}{F(k_t, \bar{e})}$$
$$\alpha_{n,t} = \frac{w_t \bar{e}}{F(k_t, \bar{e})} = \frac{F_n(k_t, \bar{e})\, \bar{e}}{F(k_t, \bar{e})}$$

Since F is constant returns to scale, factor shares sum up to 1. Now, what they look like in the steady state and over transition depends on the shape of the production function.

* Cobb-Douglass: $F(k,n) = Ak^\alpha n^{1-\alpha}$, then

$$\alpha_{k,t} = \frac{\alpha A k_t^{\alpha-1} \bar{e}^{1-\alpha} k_t}{A k_t^\alpha \bar{e}^{1-\alpha}} = \alpha, \alpha_{n,t} = 1-\alpha$$

* CES: $F(k,n) = A\left[\alpha k^\gamma + (1-\alpha) n^\gamma\right]^{\frac{1}{\gamma}}$, then

$$F_k = A\alpha k^{\gamma-1}\left[\alpha k^\gamma + (1-\alpha) n^\gamma\right]^{\frac{1}{\gamma}-1}$$
$$\alpha_{k,t} = \frac{A\alpha k_t^{\gamma-1}\left[\alpha k_t^\gamma + (1-\alpha)\bar{e}^\gamma\right]^{\frac{1}{\gamma}-1} k_t}{A\left[\alpha k_t^\gamma + (1-\alpha)\bar{e}^\gamma\right]^{\frac{1}{\gamma}}}$$
$$= \frac{\alpha k_t^\gamma}{\alpha k_t^\gamma + (1-\alpha)\bar{e}^\gamma}$$
$$\alpha_{n,t} = \frac{(1-\alpha)\bar{e}^\gamma}{\alpha k_t^\gamma + (1-\alpha)\bar{e}^\gamma}$$

  Under this specification, capital share increases during transition while labor share falls during transition. Given that factor shares seem to be stable even if we believe this model, Cobb-Douglas seems like a better choice at least for long-term analysis. It is true that labor shares - corporate labor share - have declined in some countries in the past twenty years. For more on this see the paper by Karabarbounis and Neiman (2014).

- A very common assumption: $F(K,L) = AK^\alpha L^{1-\alpha}$.

- Lucas' observation: India's GDP per capita is 1/15 times US's. What does this imply about interest rate differences? In particular, suppose we have

$$Y_{US} = AK_{US}^\alpha L_{US}^{1-\alpha}, Y_{India} = AK_{India}^\alpha L_{India}^{1-\alpha}$$
$$y_{US} = Ak_{US}^\alpha, y_{India} = Ak_{India}^\alpha$$

  where y_i is GDP per capita and k_i is capital per capita. If this model is true, then

$$\frac{r_{US}}{r_{India}} = \left(\frac{y_{US}}{y_{India}}\right)^{\frac{\alpha-1}{\alpha}} = (15)^{-2} = 1/225$$

  where we have assumed $\alpha = 1/3$. If this is true, then capital must be flowing from rich to poor countries where as in reality it does not. Why?

  - Productivity differences due to human capital
  - productivity differences due to other factors: history, language, technology
  - Taxes and other sources of government expropriation: the return is really not r_{India} as defined above but this is the before taxes/expropriation rental rate of capital
  - Maybe production functions are different: have different factor shares

- Gollin's paper: labor shares are mismeasured - not very different across countries once we account for self-employed.

- Productivity differences: human capital vs other things - evidence by Hall and Jones (1999). They assume

$$Y_i = K_i^{\alpha} \left(A_i H_i\right)^{1-\alpha}$$
$$H_i = e^{\phi(E_i)} L_i$$

  where E_i is schooling, L_i is total number of workers, and H_i is human capital. As we mentioned in class, the reason we use this specification is because we can connect it to the so-called Mincer regression that people run for the effect of schooling on wages. In particular, real wages in this model are given by

$$w_i = F_L = (1-\alpha) K_i^{\alpha} A_i^{1-\alpha} e^{(1-\alpha)\phi(E_i)} L_i^{-\alpha}$$

  taking logs we get

$$\log w_i = \text{stuff } + (1-\alpha)\,\phi\left(E_i\right)$$

  So we can run a regression of wages on years of schooling within a country and have an estimate ϕ (we can also do cross-country regression which seems more feasible as we dont need micro-data for each country)

- So we have

$$y_i = \left(\frac{K_i}{Y_i}\right)^{\frac{\alpha}{1-\alpha}} A_i h_i$$

  where h_i is human capital per capita. Notice that we have also put capital output ratio since as they argue is not susceptible to certain biases. As the figure in the slides show, doing this calculation and basically running a regression of productivity on GDP per capita gives us a coefficient of 0.6 and R-squared of 0.79. So most of the variation of GDP per capita across countries is coming from variations in productivity.

- One test of the model is on how fast is it that countries are getting closer to each other. Note that the model says that similar countries with similar productivities, discount factors and utility functions should converge to the same steady state.

- Can learn something about whether the model is correct from how fast countries converge to each other. Barro and Sala-i Martin (1992) (Cross-state regressions: 2-3% per year convergence to steady state)

$$\log\left(y_{i,t}/y_{i,t-1}\right) = \text{stuff} - \left(1-e^{-\beta}\right)\log y_{i,t-1}$$

Where is this coming from? log-linearization of the model. We will come back to this later. They show$\beta \approx 0.02$. States gets closer to each other 2% per year. Need a share of capital around 0.8!!! Is this too high? Maybe but maybe not. In particular, suppose that there are two types of capital, physical and human capital:

$$Y = AK^{\alpha} H^{\beta} L^{1-\alpha-\beta}$$

If capital and human capital have the same accumulation equation, i.e., the same depreciation rate, then I leave it to you to show that we can combine the two capitals into one hybrid capital. This capital's share in GDP is given by $\alpha + \beta$. Then a share of 0.8 is not that too far fetched.

- **Population Growth:** So given that the model does not have growth and we dont have a lot of evidence that say growth has been declining over the years, we need to create growth in some other fashion. One natural source of growth that we ignored is growth in the labor force or population. So suppose that

  $$N_t = N_0 (1+n)^t$$

  and let us normalize total leisure to 1. Then feasibility constraint is given by

  $$C_t + K_{t+1} = (1-\delta) K_t + F(K_t, N_t)$$

  Notice that I have switched to capital letters to describe aggregate allocations.

  - When there is population growth it is always tricky to write down a social welfare function. The problem is that there are new people being born every period and how do we construct social welfare function. To see this, let us index the set of people born at t by $\mathcal{I}_t$ which is some interval of length, nN_{t-1}. For simplicity we can write $\mathcal{I}_t = [N_{t-1}, N_t]$. Then the utility of an individual $i \in \mathcal{I}_t$ that is born at t, is given by

    $$U^i = \sum_{s=t}^{\infty} \beta^{s-t} u\left(c_s^i\right)$$

    where c_s^i is the per capita consumption of this individual. Now a general social welfare function as

    $$\sum_{t=0}^{\infty} \int_{\mathcal{I}_t} g_t(i) U^i di$$

    where $g_t(i)$ is the welfare weight on person i at t. Now suppose we impose that everyone's consumption is the same. Then

    $$\sum_{t=0}^{\infty} \sum_{s=t}^{\infty} \beta^{s-t} u(c_s) \int_{\mathcal{I}} g_t(i) di$$

    where c_t is per capita consumption at t. Let's call $\hat{g}_t = \int_{\mathcal{I}_t} g_t(i) di$.
    Changing the order of summation the objective function becomes

    $$\sum_{t=0}^{\infty} u(c_t) \sum_{s \leq t} \beta^{t-s} \hat{g}_s$$

Suppose we let

$$g_t(i) = \beta^t \to \hat{g}_t = \beta^t (N_t - N_{t-1})$$

where we assume $N_{-1} = 0$. Then

$$\begin{aligned}\sum_{s=0}^{t} \beta^{t-s} \hat{g}_s &= \sum_{s=0}^{t} \beta^{t-s} \beta^s (N_s - N_{s-1}) \\ &= \beta^t N_t\end{aligned}$$

so then the social welfare function becomes

$$\sum_{t=0}^{\infty} \beta^t N_t u(c_t)$$

We can also consider other social welfare functions: for example one that only puts weight on the initial generation:

$$\sum_{t=0}^{\infty} \beta^t u(c_t)$$

– Question: *If you were to solve the C.E. of this economy – with time-0 trading which is a bit weird! – what would be the welfare weights that rationalize it?*

– We can also consider other social welfare functions: for example one that only puts weight on the initial generation:

$$\sum_{t=0}^{\infty} \beta^t u(c_t)$$

This is what we discussed in class.

– Note that sometimes this can lead to time inconsistency. For example suppose that

$$g_t(i) = \beta^t \frac{1}{N_t - N_{t-1}} \to \sum_{s \leq t} \beta^{t-s} \hat{g}_s = (t+1) \beta^t$$

which is not time consistent. See the paper by Jackson and Yariv (2015).

- Let us choose the first example of the social welfare function. Then the problem of maximizing utility becomes

$$\max \sum_{t=0}^{\infty} \beta^t N_t u\left(\frac{C_t}{N_t}\right)$$

subject to

$$C_t + K_{t+1} = (1-\delta) K_t + F(K_t, N_t)$$

- Let's first get a couple of technical issues out of the way. Note that potentially things can grow without bounds. So we can quite apply the dynamic programming techniques here since our proof of principle of optimality relied on that. Second, if population growth is really fast, then we might run into trouble since the sum in the objective could be infinity. We can deal with both of these issues. As for the first one, we switch back to small letters. In other words, let

$$k_t = \frac{K_t}{N_t}, c_t = \frac{C_t}{N_t}$$

Then dividing the feasibility constraint by N_t and using CRS assumption, we have

$$\begin{aligned} c_t + \frac{K_{t+1}}{N_t} &= (1-\delta)\, k_t + F(k_t, 1) \\ c_t + \frac{N_{t+1}}{N_t}\frac{K_{t+1}}{N_{t+1}} &= (1-\delta)\, k_t + F(k_t, 1) \\ c_t + (1+n)\, k_{t+1} &= (1-\delta)\, k_t + F(k_t, 1) \end{aligned}$$

This is very similar to what we had before so presumably because of decreasing returns to scale with respect to capital we can apply the same dynamic programming techniques as before.
Now given the this modification, the objective is given by

$$N_0 \sum_{t=0}^{\infty} (\beta(1+n))^t\, u(c_t)$$

Thus, this is like a problem where the discount rate in the planning problem is $\beta(1+n)$ and impose the assumption that $\beta(1+n) < 1$. The functional equation version of it is

$$V(k) = \max_{k'} u((1-\delta)\, k + F(k, 1) - (1+n)\, k) + \beta(1+n)\, V(k')$$

This problem is not very different than the old problem. In particular, its Euler equation is given by

$$(1+n)\, u_{ct} = \beta(1+n)\left[F_k(k_{t+1}, 1) + 1 - \delta\right] u_{ct+1}$$

which is identical to the one we had before except in terms of per-capita stuff - capital and consumption.

- How are things look like in the long-run? They look like something that we referred to as balanced growth path, all per capita variables growing at the same rate. Note that since we have decreasing returns to scale with respect to capital-per-capita, again the capital-per-capita converges to a level k^* which is given by

$$\beta\left[F_k(k^*, 1) + 1 - \delta\right] = 1 \tag{6}$$

All of this implies that growth of per capita variables in the BGP - balanced growth path - is 0. So again we can generate long growth.

- **Productivity Growth.** The last place to put growth in is productivity. It turns out there are many ways to do this:

– **Labor Augmenting Productivity Growth or Harrod-Neutral:**

$$Y_t = F(K_t, A_t L_t)$$

where $A_t = A_0 (1+g)^t$ and $N_t = (1+n)^t N_0$. A balanced growth path for this economy is one where all per capita variables grow at rate $\hat{g}$. We must have

$$(1+\hat{g})^t (1+n)^t Y_0 = F\left(K_0 (1+\hat{g})^t (1+n)^t, (1+g)^t A_0 N_0 (1+n)^t\right)$$

Thus the economy can potentially grow in the long run. The question is how can we show that.

* We are going to assume that

$$u(c) = \begin{cases} \frac{c^{1-\sigma}}{1-\sigma} & \sigma \neq 1 \\ \log c & \sigma = 1 \end{cases}$$

We also define the following variable

$$X_t = A_t N_t$$

We also define

$$\hat{c}_t = \frac{C_t}{X_t}, \hat{k}_t = \frac{K_t}{X_t}$$

Then the feasibility constraint becomes

$$\hat{c}_t + (1+n)(1+g)\hat{k}_{t+1} = \hat{k}_t (1-\delta) + F\left(\hat{k}_t, 1\right)$$

We can also write the objective as

$$\sum_{t=0}^{\infty} \left(\beta (1+n)(1+\hat{g})^{1-\sigma}\right)^t u(\hat{c}_t)$$

Then the Bellman equation associated with this is

$$V\left(\hat{k}\right) = \max_{\hat{k}'} u\left((1-\delta)\hat{k} + F\left(\hat{k}, 1\right) - (1+n)(1+\hat{g})\hat{k}\right) + \beta (1+n)(1+\hat{g})^{1-\sigma} V\left(\hat{k}'\right)$$

The Euler equation is given by

$$u'(\hat{c}_t) = \beta (1+g)^{-\sigma} \left[1 - \delta + F_{k,t+1}\left(\hat{k}_{t+1}, 1\right)\right] u'(\hat{c}_{t+1})$$

As before we can show that the optimal allocations in this economy converge:

$$\lim_{t\to\infty} \hat{k}_t = \hat{k}^*$$

where

$$1 = \beta (1+g)^{-\sigma} \left[1 - \delta + F_k\left(\hat{k}^*, 1\right)\right]$$

* Thus a unique balanced growth path exists and the economy always converges to that. The growth rate is approximately given by$g$. Now, let's see what happens to other Kaldor facts, we have

$$\begin{aligned} w_t &= A_t F_N\left(K_t, A_t N_t\right) \text{ firms' first order condition with } w_t \text{ :the real wage} \\ &= A_t F_N\left(\frac{K_t}{A_t N_t}, 1\right) \text{ since } F \text{ is homogenous of degree 1 - show this!} \\ &= A_t F_N\left(\hat{k}^*, 1\right) \end{aligned}$$

Note that the above holds on a balanced growth path. So wages also grow at a constant rate.

$$\begin{aligned} r_t = F_K\left(K_t, A_t N_t\right) &= F_K\left(\frac{K_t}{A_t N_t}, 1\right) \\ &= F_K\left(\hat{k}^*, 1\right) \end{aligned}$$

which is constant. Note that BGP level of capital-output ratio in this economy is given by

$$\begin{aligned} \frac{K_t}{Y_t} = \frac{\hat{k}^* A_t N_t}{F\left(K_t, A_t N_t\right)} &= \frac{\hat{k}^*}{F\left(\frac{K_t}{A_t N_t}, 1\right)} \\ &= \frac{\hat{k}^*}{F\left(\hat{k}^*, 1\right)} \end{aligned}$$

In the models without long-run growth capital -output ratio is given by

$$\frac{k^*}{F\left(k^*, 1\right)}$$

where k^* satisfies (6). Note that when $g > 0$, then $k^* > \hat{k}^*$ and as a result, capital-output ratio is lower in the economy with growth. Intuitively, since in this economy consumption is growing people will have a lower marginal utility of consumption tomorrow relative to today. As a result, the rate of return on capital should increase which means an increase in capital-output ratio. Therefore, the steady state level of capital-output ratio depends on the growth rate of the economy.

* This is basically our first model that can match the Kaldor facts and growth is all determined by productivity growth. It is somewhat disappointing however as growth does not depend on anything like taxes/property rights; economic decision makings, etc. Is this a bad result? Perhaps. It is because we see a lot of countries where property rights are not enforced and governments expropriate stuff and they are not growing very fast. *Question:* Can the causality be reversed?

- **Hicks neutral productivity growth**:

$$Y_t = A_t F(K_t, N_t)$$

For Cobb-Douglass – $F(K,N) = K^\alpha N^{1-\alpha}$ – this is equivalent to labor-augmenting productivity growth while the BGP growth rate has to be adjusted. In particular, we can show that the growth rate of the BGP in this case is $(1+g)^{\frac{1}{1-\alpha}} - 1$ – *Show this!* Therefore the long-run growth rate of the economy depends on the share of labor and capital in the production function.

- **Capital Augmenting Productivity Growth:**

$$Y_t = F(A_t K_t, L_t)$$

again with Cobb-douglas it is similar to what we have before. We also have a stronger result due to Uzawa (1961):

- Uzawa's theorem - much stronger statements are also true (see paper by Grossman et al. (2016)):

- Suppose that $Y = F(A_t K, L)$ where $\frac{A_{t+1}}{A_t} = 1+g$, then the only way to have balanced growth is to have $F(K,L) = \hat{A} K^\alpha L^{1-\alpha}$.

• This is a problem: How can we think of falling labor share, changes in investment good prices, etc?

• **The AK model** a la Rebelo (1991)

- Suppose that instead of the usual production function, there is only one factor of production and that is capital. In other words,

$$Y_t = AK_t$$

The question is what happens to growth in this economy. We will again assume that the utility function belongs to the standard CRRA class of utility functions. Then the planning problem associated with this economy is

$$V(K_0) = \max \sum_{t=0}^{\infty} \beta^t u(C_t)$$

subject to

$$C_t + K_{t+1} = (A + 1 - \delta) K_t$$
$$K_0 : \text{given}$$

Note that this problem is kind of homogeneous in K_0. In other words, if we multiply K_0 by a factor of λ, the solution of the problem will also be multiplied by λ (you are asked to prove this in a homework). As a result and because of the form of the utility function, we can show

$$V(K_0) = B \frac{K_0^{1-\sigma}}{1-\sigma}$$

for some constant B. Note again that we cannot use the very standard techniques of dynamic programming (SLP chapter 4) to solve this problem. See Alvarez and Stokey (1998) for ways of how to deal with this problem. Once we know the value function has this form, we can write

$$B\frac{K^{1-\sigma}}{1-\sigma} = \max_{C,K} \frac{C^{1-\sigma}}{1-\sigma} + \beta B \frac{(K')^{1-\sigma}}{1-\sigma}$$

subject to

$$C + K' = (A + 1 - \delta) K$$

The solution to the above can be simply found using FOCs:

$$\frac{C^{-\sigma}}{\beta B (K')^{-\sigma}} = 1 \to K' = C (\beta B)^{1/\sigma}$$

$$K' = \frac{(\beta B)^{1/\sigma}}{(\beta B)^{1/\sigma} + 1} (A + 1 - \delta) K$$

$$C = \frac{1}{(\beta B)^{1/\sigma} + 1} (A + 1 - \delta) K$$

This means that

$$\frac{Y_1}{Y_0} = \frac{K_1}{K_0} = \frac{(\beta B)^{1/\sigma}}{(\beta B)^{1/\sigma} + 1} (A + 1 - \delta) = 1 + \hat{g}$$

The same property holds for C_1 relative to C_0. This means that the economy converges to its balanced growth path right away. In other words, there are no transition periods. Now in order to find the growth rate of the economy, consider the standard Euler Equation

$$C_t^{-\sigma} = \beta (A + 1 - \delta) C_{t+1}^{-\sigma} \to \frac{C_{t+1}}{C_t} = [\beta (A + 1 - \delta)]^{\frac{1}{\sigma}}$$

Now as we see the growth rate of the economy is now tied to the parameters of preferences and technology. In this sense, this is an endogenous growth model. It has some important implications though. It means that similar countries, do not converge - which seems to be there in the data (see Mankiw et al. (1992)) It also says that countries that have a higher productivity grow faster. That is perhaps not true given how correlated the level of productivity is with GDP per capita and specially because rich countries are not growing very fast. That does not mean that the model does not work because we can still play around with other parameters, risk aversion or discount rate and can create differences in growth rates across countries. As you will also see government policy can contribute to the variations in growth rate.

– The AK model has a weird production function as there is no factor of production other than capital. All income is accrued to owners of capital. This seems obviously wrong! It however is possible to allow for other types of income and get a similar result. The key is that all factors must be accumulated and we must have constant returns to scale.

- **The AKH model**
  - Suppose now that production is done using physical and human capital
$$Y_t = AK_t^{\alpha} H_t^{1-\alpha}$$
The key question is how to specify the accumulation process for human capital. In particular, what is the cost of human capital accumulation and how does human capital grow over time. As in the case of physical capital, we can specify an accumulation equation for evolution of human capital over time
$$H_t = H_{t-1}(1-\delta_h) + X_{t,h}$$
where $X_{t,h}$ is investment in human capital. There are various ways to think about what is the nature of $X_{t,h}$. It could be the infrastructure allocated to universities and schools and other learning facilities, i.e., the cost is in terms of consumption good. Alternatively, it is possible that people actually spend time to learn new stuff in which case the cost is in terms of leisure. In a model with inelastic labor supply, this would mean that people can give up earning wages today and instead invest in their human capital and earn income in the future. Let S_t be total time spent on accumulation of human capital and Z_t be the resources allocated to human capital investment, then we must have
$$\begin{aligned} Z_t + X_t + C_t &= F(K_t, H_t - S_t) \\ X_{t,h} &= g(S_t, Z_t) \\ K_{t+1} &= X_{t,k} + (1-\delta_k) K_t \\ H_{t+1} &= X_{t,h} + (1-\delta_h) H_t \end{aligned}$$
where g is some function that combines time and resources spent on human capital accumulation.
  - This will be greatly simplified when we assume that g does not depending on S and is linear in Z_t . In other words, suppose that
$$g(S_t, Z_t) = Z_t$$
In addition, to make life easy, suppose that $\delta_h = \delta_k = \delta$. Then we can write the optimization problem as follows:
$$\max \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\sigma}}{1-\sigma}$$
subject to
$$\begin{aligned} C_t + X_{h,t} + X_{k,t} &= F(K_t, H_t) \\ K_{t+1} &= X_{t,k} + (1-\delta) K_t \\ H_{t+1} &= X_{t,h} + (1-\delta) H_t \\ K_0, H_0&\text{: given} \end{aligned}$$

Now, we can write this problem recursively but let's try to get an insight by thinking about the Euler equation. Note that in this model, we have two intertemporal Euler Equation, one for each of the accumulated factors:

$$C_t^{-\sigma} = \beta \left(F_{K,t+1} + 1 - \delta \right) C_{t+1}^{-\sigma}$$
$$C_t^{-\sigma} = \beta \left(F_{H,t+1} + 1 - \delta \right) C_{t+1}^{-\sigma}$$

This implies that

$$F_{K,t+1} = F_{H,t+1} \rightarrow \alpha \left(\frac{H_{t+1}}{K_{t+1}} \right)^{1-\alpha} = (1-\alpha) \left(\frac{K_{t+1}}{H_{t+1}} \right)^{\alpha}$$

Note that the above relationship is a simple no-arbitrage condition between the returns on the two assets - human and physical capital. This implies that

$$\frac{K_{t+1}}{H_{t+1}} = \frac{\alpha}{1-\alpha}$$

and as a result, the return to capital is given by

$$F_{K,t+1} = \alpha A \left(\frac{H_{t+1}}{K_{t+1}} \right)^{1-\alpha} = \alpha \left(\frac{1-\alpha}{\alpha} \right)^{1-\alpha} A = \alpha^{\alpha} \left(1-\alpha \right)^{1-\alpha} A$$

Therefore, from the Euler equation

$$\frac{C_{t+1}}{C_t} = \left(\beta \left(\alpha^{\alpha} \left(1-\alpha \right)^{1-\alpha} A + 1 - \delta \right) \right)^{\frac{1}{\sigma}}$$

That is the growth rate of this economy is always constant and as is given by the above expression. This is basically identical to the AK model except that the factors of production are the traditional ones - labor and capital where labor is quality adjusted and this quality can grow over time in an endogenous way. Note that unlike the AK model, this model exhibits some transition as the K/H ratio at time 0 might not be the same as $\frac{\alpha}{1-\alpha}$. However, this transition is quite fast and will occur in one period.

- *Question: what happens when $\delta_h \neq \delta_k$*
- The above model illustrates that we can have growth endogenously determined if all factors can be accumulated and we have CRS. In other words, there are no fixed factors. Is this a good assumption? probably not. One can think of a few fixed factors:
  * Time: Recall the one-sector growth model where total endowment of leisure is fixed. In this model, labor is a fixed factor. Note that this is true even if we have population growth and population growth is exogenous. But maybe population is also an endogenous factor of production. People clearly decide how many kids to have and this decision can potentially be affected by economic variables. For example, one can think of the decline in fertility as being correlated with increase in women's participation in the labor force. This means that maybe we can even treat population as an accumulated factor and not a fixed factor. This is the subject of a series of very interesting papers by Barro and Becker (1989) and other papers by my advisor Larry E. Jones and various coauthors (including yours truly!).

    * Land: As long as we do not go to other planets or utilize the oceans and the sky, we need land as a factor of production and its level is fixed. Once we have a fixed factor and CRS, we are back to our one-sector growth model.
    * Natural resources: Oil or more generally natural resources are an important part of production these days. It is hard to imagine that we have unlimited amount of oil and even if so that we can invest and create oil out of thin air.
  - The above discussion is kind of disappointing as perhaps growth shouldn't be stopped by inclusion of fixed factors since we have had these fixed factors for a long time yet we have been growing for a long time. For example, keep in mind that the price of oil has not really increased over all these years. If we were living a world with fixed factor and CRS, over the course of transition we should see an increase in price of oil as output increases over time.

Growth in Continuous Time

- In the growth literature, there is a tradition of using continuous time models since we can take derivatives with respect to time in order to calculate the growth rates. This makes life sometime easier. In order to familiarize you with this approach, I am going to describe to you how we can solve the standard one-sector growth model in continuous time.

- The intuitive way to think about this is to think about a discrete time model where the length of time horizon is shrinking to 0 . In particular, think of our model and suppose that the length of time interval is Δ. Furthermore, rewrite the flow variables so that they are in terms of stuff per unit of time. So $\Delta \times c_t$ is total consumption over an interval of length Δ and thus c_t is now consumption per unit of time. If we take the standard one sector growth model, we can write the feasibility constraint as

$$\Delta \cdot c_t + k_{t+\Delta} = k_t \left(1 - \delta \cdot \Delta\right) + \Delta \cdot f\left(k_t\right)$$
$$\frac{k_{t+\Delta} - k_t}{\Delta} = f\left(k_t\right) - c_t - \delta k_t$$

  Note that since k_t is a stock variable, we should not multiply it by Δ. However, since depreciation and output are both flow variables, their level shrink with the length of time and thus we have the above equation. Now as we send Δ to 0, the above becomes

$$\dot{k}_t = \lim_{\Delta \to 0} \frac{k_{t+\Delta} - k_t}{\Delta} = f\left(k_t\right) - \delta k_t - c_t$$

  Equivalently, the utility of the household is given by

$$\int_0^\infty e^{-\rho t} u\left(c_t\right) dt$$

  Thus the optimal control problem is given by

$$\max_{c_t, k_t} \int_0^\infty e^{-\rho t} u\left(c_t\right) dt$$

subject to

$$\dot{k}_t = f(k_t) - \delta k_t - c_t$$
$$k_0 \text{: given}$$

Let's start from a case where the time-horizon is finite and is equal to T. Here I will heuristically derive the optimum. You can check Daron's textbook for a more formal illustration. From theorems in Luenberger (1969), it can be shown that the solution to this must also maximize the lagrangian given by

$$\mathcal{L} = \int_0^T e^{-\rho t} u(c_t)\, dt + \int_0^T \lambda_t \left[f(k_t) - \delta k_t - c_t - \dot{k}_t \right] dt$$

We can use integration by parts to write

$$\begin{aligned} \int_0^T \lambda_t \dot{k}_t dt &= \int_0^T \lambda_t d(k_t) \\ &= \lambda_t k_t |_0^T - \int_0^T k_t \dot{\lambda}_t dt \\ &= \lambda_T k_T - \lambda_0 k_0 - \int_0^T k_t \dot{\lambda}_t dt \end{aligned}$$

Thus we can right the Lagrangian as

$$\mathcal{L} = \int_0^T e^{-\rho t} u(c_t)\, dt + \int_0^T \left[\lambda_t (f(k_t) - \delta k_t - c_t) + k_t \dot{\lambda}_t \right] dt + \lambda_0 k_0 - \lambda_T k_T$$

Now the FOCs are given by

$$\begin{aligned} e^{-\rho t} u'(c_t) &= \lambda_t \\ \lambda_t (f'(k_t) - \delta) + \dot{\lambda}_t = 0 \Rightarrow -\frac{\dot{\lambda}_t}{\lambda_t} &= f'(k_t) - \delta \\ \rho - \frac{u''(c_t) c_t}{u'(c_t)} \frac{\dot{c}_t}{c_t} &= f'(k_t) - \delta \\ \sigma(c_t) \frac{\dot{c}_t}{c_t} &= f'(k_t) - \delta - \rho \end{aligned}$$

The above is the continuous time version of the Euler equation. Note that optimality also requires

$$\lambda_T k_T = 0$$

if consumption is always finite , then $\lambda_T > 0$ which means $k_T = 0$. Now, when $T = \infty$, then the same Euler equation holds yet the above condition becomes

$$\lim_{t \to \infty} \lambda_t k_t = 0 \text{ or } \lim_{t \to \infty} e^{-\rho t} u'(c_t) k_t = 0$$

This is referred to as the transversality condition.

- We can also formulate this continuous time version of the growth model recursively. In particular, suppose that we define the value function associated with the above sequence problem as follows:
$$v(k_0) = \max \int_0^\infty e^{-\rho t} u(c_t)\, dt$$
subject to
$$\dot{k}_t = f(k_t) - c_t - \delta k_t$$
I do this heuristically. For more rigurous proofs I refer you to Kushner and Dupuis (2013)

- For any variable Δ, we can write
$$v(k_0) = \max \int_0^\Delta e^{-\rho t} u(c_t)\, dt + e^{-\rho\Delta} v(k_\Delta)$$
subject to
$$\dot{k}_t = f(k_t) - c_t - \delta k_t$$

- We can rewrite the above as
$$\begin{aligned}
0 &= \max \frac{\int_0^\Delta e^{-\rho t} u(c_t)\, dt}{\Delta} + \frac{e^{-\rho\Delta} v(k_\Delta) - v(k_0)}{\Delta} \\
0 &= \lim_{\Delta\to 0} \max \frac{\int_0^\Delta e^{-\rho t} u(c_t)\, dt}{\Delta} + \frac{e^{-\rho\Delta} v(k_\Delta) - v(k_0)}{\Delta} \\
0 &= \max u(c_0) + \frac{d}{d\Delta}\left(e^{-\rho\Delta} v(k_\Delta)\right)\Big|_{\Delta=0} \\
0 &= \max_{c_0} u(c_0) + v'(k_0)\, \dot{k}_0 - \rho v(k_0) \\
\rho v(k_0) &= \max_{c} u(c) + v'(k_0)\left[f(k_0) - c - \delta k_0\right]
\end{aligned}$$
The above functional equation is often referred to as the Hamilton-Jacobi-Bellman or HJB equation. The FOC of the above optimization problem is given by
$$u'(c) = v'(k_0) \to c = (u')^{-1}(v'(k_0))$$
Hence, the functional equation above becomes
$$\rho v(k) = u\left((u')^{-1}(v'(k))\right) + v'(k)\left[f(k) - (u')^{-1}(v'(k)) - \delta k\right]$$
The above is a differential equation which can be solved by various methods that have been developed over time. The goal of this part was to make you familiar with continuous time techniques. Later when we discuss search models we will come back to them.

Growth with externalities a la Romer (1986):

- One idea to overturn the above disappointment is to think about relaxing some of those assumptions, for example CRS assumption. In other words, maybe we have fixed factors but we have increasing returns to scale. This idea goes back to Arrow and Romer formalized it beautifully in his 1986 paper. So, what is the idea here? It is that accumulation of capital or human capital etc, can be thought of as creating ideas that can be used by others without any cost, i.e., these ideas are non-rivalrous. In other words, at the firm level there is CRS but once we aggregate things up we will have increasing returns to scale. Note also that this is necessarily going to imply that there is externality since firm's capital placement decisions affect other firms.

- To formalize this let us consider the following adjustment of the standard one-sector growth model on the production side. Suppose there is a continuum of firms producing the single consumption good using labor and capital that are indexed by $i \in [0, 1]$. Each of these firms have a production function given by

$$y_{t,i} = (k_{t,i})^{\alpha} (A_t n_{t,i})^{1-\alpha}$$

where

$$A_t = B \int_0^1 k_{t,i} di$$

The rest of the model is as before. Note that in this model, because we have an externality, the solution of the CE is not Pareto optimal. So we just solve for the C.E. Each firm's optimization problem is given by

$$\max (k_{t,i})^{\alpha} (A_t n_{t,i})^{1-\alpha} - r_t k_{t,i} - w_t n_{t,i}$$

The FOCs imply that

$$\alpha \left(\frac{k_{t,i}}{A_t n_{t,i}} \right)^{\alpha-1} = r_t \rightarrow \frac{k_{t,i}}{n_{t,i}} = \frac{k_{t,j}}{n_{t,j}}, \forall i, j \in [0, 1]$$

Thus, if aggregate capital stock in the economy is K_t and total labor is L, then

$$\frac{k_{t,i}}{n_{t,i}} = \frac{K_t}{L} \rightarrow r_t = \alpha \left(\frac{A_t n_{t,i}}{k_{t,i}} \right)^{1-\alpha} = \alpha \left(\frac{A_t L}{K_t} \right)^{1-\alpha} = \alpha \left(\frac{B K_t L}{K_t} \right)^{1-\alpha} = \alpha (BL)^{1-\alpha}$$

and

$$w_t = (1-\alpha) A_t^{1-\alpha} \left(\frac{k_{t,i}}{n_{t,i}} \right)^{\alpha} = (1-\alpha) (BK_t)^{1-\alpha} \left(\frac{K_t}{L} \right)^{\alpha} = \frac{(1-\alpha) (BL)^{1-\alpha} K_t}{L}$$

Total production is also given by

$$\int_0^1 y_{t,i} di = \int_0^1 (k_{t,i})^{\alpha} (A_t n_{t,i})^{1-\alpha} di = A_t^{1-\alpha} \int_0^1 k_{t,i} \left(\frac{L}{K_t} \right)^{1-\alpha} di$$
$$= (BK_t)^{1-\alpha} K_t \frac{L^{1-\alpha}}{K_t^{1-\alpha}} = (BL)^{1-\alpha} K_t$$

Now, we have to finish this by solving the household side and find market clearing prices:

$$\max \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\sigma}}{1-\sigma}$$

subject to

$$\sum_{t=0}^{\infty} p_t \left[C_t + X_t\right] \leq \sum_{t=0}^{\infty} p_t \left[r_t K_t + w_t L\right]$$
$$K_{t+1} = (1-\delta) K_t + X_t$$

We must have that

$$p_t = p_{t+1} \left[r_{t+1} + 1 - \delta\right]$$
$$\beta^t C_t^{-\sigma} = \lambda p_t \Rightarrow C_t^{-\sigma} = \beta C_{t+1}^{-\sigma} \left(r_{t+1} + 1 - \delta\right)$$
$$\frac{C_{t+1}}{C_t} = \left(\beta \left(\alpha \left(BL\right)^{1-\alpha} + 1 - \delta\right)\right)^{\frac{1}{\sigma}}$$

As we see the model basically acts like an AK model with an endogenous growth rate.

- Scale effect: a property that can be seen above is that L or population (or any other fixed factor that you can think of: land, natural resources etc.) causes a higher growth. Is this a reasonable implication? perhaps for 19th century but not for twentieth century. In part because in 19th century, the world saw large increases in population and at the same time persistent growth. On the other hand, in twentieth century population growth has been slowing down and yet no particularly strong sign of slow down in growth. This has led people to write down models that does not have the scale effect. See series of papers by Chad Jones in the 90's for explaining this or by Kortum or Segerstorm.

- Despite their differences the above model and the AK model presented before have a lot in common. They are both linear. This linearity together with homotheticity of the preferences make them very tractable but also has stark implications: scale effect, divergence, positive correlation of TFP and growth etc, are all part of these strong predictions of such model. Anything that wants to break these needs to add some concavity in the model.

- As a side note, in your homework you will show that the model presented above is actually inefficient. This is because when firms invest, they do not internalize the effect that they will have on other firms. Typically in these situations, a tax or subsidy can restore efficiency. Which one should be used?

Endogenous Growth through Product Innovation.

- We are ready to finish our discussion of growth models by considering models of innovation. There is a sense in which in every model that we have discussed above, there is no decision regarding increases in productivity. Romer's externality model has capital growing yet no-one really is making a decision about how to grow. In this section, I will show you a model where people decide to innovate and ideas are non-rivalrous. This is based on Romer (1990).

- Before starting to setup the model, note the following observation: if innovation is costly and ideas can be copied with zero cost then we cannot have perfect competition in product markets and equilibrium innovation at the same time. To see this, consider a two-stage model where a bunch of innovators innovate and lower the variable cost of production by paying a fixed cost in the first stage. In the second stage, these innovators compete a la Bertrand. This would imply that in the second stage, prices are equal to marginal cost or economic profits are zero. Now since innovation is costly, it must be that innovators do not have an incentive to innovate. Thus the only equilibrium is one in which no one innovates.

- The above logic implies that we cannot use what we have been using all along - CE. We need to allow for some sort of monopoly power. Oligopoly models are typically very non-linear and could potentially make life hard. Thankfully, Dixit and Stiglitz (1977) came up with their very tractable model of imperfect competition which allows us to keep the analysis tractable - as we will see below.

- **Monopolistic Competition a la Dixit and Stiglitz (1977):** Consider the following model of imperfect competition. There is a unit continuum of goods indexed by $i \in [0, 1]$ produced by only one producer - so each good is a differentiated product from the others. These producers produce these goods using labor only and their production function is given by $y = An$. They understand that there are the only producers of each variety. There is a single unit of households purchasing a bundle of these goods using their labor income which they earn by supplying labor to the firms.

- Households problem is given by

$$\max_{\{x(i)\}} \left(\int_0^1 x(i)^{1-\frac{1}{\sigma}} \, di \right)^{\frac{\sigma}{\sigma-1}}$$

subject to

$$\int_0^1 p(i) \, x(i) \, di \leq 1 + \Pi$$

where Π is the total profits by firms. Let the solution to the above be given by $\{x(i; \{p(i)\}, w)\}_{i\in[0,1]}$. Note that here we have normalized wage, w, to 1 - since we can always do this to one price.

- Given the demand function from households, firm i solves

$$\max_{n(i), p(i)} p(i) \, An(i) - n(i)$$

subject to

$$x(i; \{p(i)\}, w) = An(i)$$

Thus an equilibrium in this economy is described by $\{x(i), p(i), n(i)\}_{i\in[0,1]}$ where allocations and prices solve the above problems and markets clear

$$x(i) = An(i)$$

$$\int n(i) \, di = 1$$

- The solution to the household's problem satisfies the following first order condition

$$x(i)^{-\frac{1}{\sigma}}\left(\int_0^1 x(i)^{1-\frac{1}{\sigma}}\,di\right)^{\frac{1}{\sigma-1}} = \lambda p(i) \rightarrow x(i) = \lambda^{-\sigma} p(i)^{-\sigma}\left(\int_0^1 x(i)^{1-\frac{1}{\sigma}}\,di\right)^{\frac{\sigma}{\sigma-1}}$$

replacing in the budget constraint

$$\begin{aligned} 1+\Pi &= \int_0^1 p(i)\,x(i)\,di = \lambda^{-\sigma}\int_0^1 p(i)^{1-\sigma}\left(\int_0^1 x(i)^{1-\frac{1}{\sigma}}\,di\right)^{\frac{\sigma}{\sigma-1}} di \\ &\rightarrow \lambda^{-\sigma}\left(\int_0^1 x(i)^{1-\frac{1}{\sigma}}\,di\right)^{\frac{\sigma}{\sigma-1}}\int_0^1 p(i)^{1-\sigma}\,di = 1+\Pi \\ x(i) &= \frac{1+\Pi}{\int_0^1 p(i)^{1-\sigma}\,di}p(i)^{-\sigma} \end{aligned}$$

- Given the above demand function, each firm's problem is given by

$$\max_{p(i),n} p(i)\,An - n$$

subject to

$$\frac{1+\Pi}{\int_0^1 p(i)^{1-\sigma}\,di}p(i)^{-\sigma} = An$$

We can eliminate $p(i)$ in the above optimization and write it as

$$\max_n (An)^{1-\frac{1}{\sigma}}\left(\frac{1+\Pi}{\int_0^1 p(i)^{1-\sigma}\,di}\right)^{\frac{1}{\sigma}} - n$$

Taking first order condition

$$\left(1-\frac{1}{\sigma}\right)A^{1-\frac{1}{\sigma}}n^{-\frac{1}{\sigma}}\left(\frac{1+\Pi}{\int_0^1 p(i)^{1-\sigma}\,di}\right)^{\frac{1}{\sigma}} = 1 \rightarrow \left(1-\frac{1}{\sigma}\right)Ap(i) = 1 \rightarrow p(i) = \frac{\sigma}{\sigma-1}\frac{1}{A}$$

The above shows that the optimal price for the monopolist is marginal cost A^{-1} times one plus the markup $\frac{1}{\sigma-1}$. Note that in the above σ Since markups are higher than 1, then the monopolist earns some rent. Note that when we calculated the above FOC's we did not take into account the fact that if monopolist i changes price $p(i)$, it does not change the integral $P^{1-\sigma} = \int_0^1 p(i)^{1-\sigma}\,di$. This is because each monopolist is small and only takes into account the change in demand caused by the change in his/her own price. If instead we wrote a model with a discrete number of goods, then we would have take this into account since each monopolist is large.

- The above equations lead to the following quantities

$$x(i) = x = \frac{1+\Pi}{P^{1-\sigma}}p(i)^{-\sigma} = \frac{1+\Pi}{P} = A$$

Thus total demand for labor is given by

$$\int_0^1 n(i)\,di = \int_0^1 \frac{x(i)}{A}di = 1$$

Thus all the market clearings are satisfied.

- The above model is basic model that allows for imperfect competition and yet it is tractable enough to try to embed in a macro model. In fact, much of monetary economics, growth theory, international trade, etc. use the framework above to think about lack of competition in aggregate models of the economy.

- Now, we can setup our model of growth through innovation. Suppose that final output is produced using a bundle of intermediate goods and labor:

$$Y_t = \int_0^{N_t} x_t(i)^\alpha \, di L^{1-\alpha}$$

where i is associated with an intermediate good that is produced by an intermediate good firm.

- Each intermediate good is produced using a technology that uses the final good as its only input. In particular, producing a unit of good i requires ψ unit of the final good.

- Innovation is coming up with a new intermediate good - this is somewhat referred to as process innovation as opposed to product innovation which is about innovations in the final good producing sector (you can check out Daron's textbook where he shows that the two could be equivalent). It is done as follows: in each period some people decide whether to pay a fixed cost κ to innovate - this is in terms of the final good. The outcome of innovation is random: with probability $\eta < 1$ it succeeds and there will be a new good. With a complementary probability this does not work out. Thus, the decision to innovate or not, depends on the discounted payoff of coming up with an intermediate good i:

$$V_t(i) = \int_t^\infty e^{-\int_t^s r_\tau d\tau} \pi_s(i)\,ds$$

where $\pi_s(i)$ is the flow value of profits for blueprint associated with good i in period s. Note that we have used accumulated interest rate for the way the future profits are discounted. The idea is that households own these firms and they demand an interest rate r_t in period t. Note that $V_t(i)$ satisfies

$$\begin{aligned}\dot{V}_t(i) &= -\pi_t(i) + \int_t^\infty r_t e^{-\int_t^s r_\tau d\tau} \pi_s(i)\,ds \\ &= -\pi_t(i) + r_t V_t(i)\end{aligned}$$

or

$$r_t V_t(i) = \pi_t(i) + \dot{V}_t(i)$$

The above equation is intuitive. The return paid on innovation is equal to profits plus the change in valuations in the future.

- We let the total cost spent on innovation be given by Z_t units of the final good. Since η fraction of these will be successful, we must have that

$$\dot{N}_t = \eta Z_t$$

Thus resource constraint for the final good is given by

$$\kappa Z_t + \psi X_t + C_t = Y_t$$

where

$$X_t = \int_0^{N_t} x_t(i)\, di$$

- Households preferences are given by

$$\int_0^{\infty} e^{-\rho t} \frac{C_t^{1-\sigma}}{1-\sigma} dt$$

They get labor income and invest in intermediate good firms. In particular, we can think of a financial intermediary that pulls funds from the households and lends to owners of the blueprints. In return, it will give an interest rate of r_t to the households. As a result, the households Euler equation is given by

$$\sigma \frac{\dot{C}_t}{C_t} = r_t - \rho$$

- Now an equilibrium of this economy is given by sequence of prices, $r_t, w_t, p_t(i)$ and allocations $x_t(i), C_t, Z_t, N_t, Y_t$ such that households maximize given these prices; firms maximize - both in terms of their decision to innovate and their decision to produce; and markets clear (final goods, labor, intermediate goods).

- The interesting parts of this model are on the firm side. The final good producer solves the following:

$$\max_{L_t, x_t(i)} \left(\int_0^{N_t} x_t(i)^{\alpha} di \right) L_t^{1-\alpha} - \int_0^{N_t} p_t(i)\, x_t(i)\, di - w_t L_t$$

Optimality conditions for this problem are given by

$$\alpha x_t(i)^{\alpha-1} L_t^{1-\alpha} = p_t(i) \tag{7}$$

$$(1-\alpha) \left(\int_0^{N_t} x_t(i)^{\alpha} di \right) L_t^{-\alpha} = w_t$$

- The intermediate good producers problem has two parts: 1. the production decision conditional on being successful in innovating, 2. decision to innovate. In each period, the above FOCs give the demand curve for i and thus the problem of finding quantity of intermediate good i at t is given by

$$\max p_t(i)\, x_t(i) - \psi x_t(i)$$

subject to (7). This as before implies that

$$\alpha p_t(i) = \psi \rightarrow p_t(i) = \frac{\psi}{\alpha}$$

Since these prices are the same for all intermediate goods, the quantities follow the same property and thus we have $x_t(i) = x_t$ with

$$\alpha x_t^{\alpha-1} L^{1-\alpha} = \frac{\psi}{\alpha} \rightarrow x_t = \left(\frac{\alpha^2}{\psi}\right)^{\frac{1}{1-\alpha}} L$$

$$(1-\alpha) N_t x_t^{\alpha} L^{-\alpha} = w_t \rightarrow w_t = N_t (1-\alpha) \left(\frac{\alpha^2}{\psi}\right)^{\frac{\alpha}{1-\alpha}}$$

Given these conditions, profits of the intermediate good producers are given by

$$\pi_t(i) = p_t(i) x_t(i) - \psi x_t(i) = \frac{1-\alpha}{\alpha} \left(\frac{\alpha^2}{\psi}\right)^{\frac{1}{1-\alpha}} L = \pi$$

- Now, let us consider the decision to innovate. On the margin and at the optimum, innovators must be indifferent between innovating and not innovating

$$\eta V_t(i) = \kappa$$

Note that our modeling of the above free-entry condition is somewhat reduced form. The idea is that if the above relationship is not violated either no-one would like to innovate or that everyone wants to innovate.

- Let us conjecture a BGP where $r_t = r^*$ is constant and the growth rate of the economy is constant as well. We have

$$V_t(i) = \int_t^{\infty} e^{-r^*(s-t)} \pi ds = \frac{1}{r^*}\pi = \frac{1}{r^*}\frac{1-\alpha}{\alpha} \left(\frac{\alpha^2}{\psi}\right)^{\frac{1}{1-\alpha}} L = \frac{\kappa}{\eta} \rightarrow r^* = \frac{\eta}{\kappa}\frac{1-\alpha}{\alpha} \left(\frac{\alpha^2}{\psi}\right)^{\frac{1}{1-\alpha}} L$$

Thus the growth rate of consumption is given by

$$g_c = \frac{\dot{C}_t}{C_t} = \frac{1}{\theta}(r^* - \rho) = \frac{1}{\theta}\left(\frac{\eta}{\kappa}\frac{1-\alpha}{\alpha} \left(\frac{\alpha^2}{\psi}\right)^{\frac{1}{1-\alpha}} L - \rho\right)$$

The above pins down the growth rate of the economy. It is related to the fixed cost of innovation, the probability of success, elasticity of substitution and the size of the economy L. As before, the scale effect is present in this model. An increase in the size of the economy leads to an increase in the growth rate of the economy.

- Note also that output in this economy is given by

$$Y_t = N_t \left(\frac{\alpha^2}{\psi}\right)^{\frac{\alpha}{1-\alpha}} L$$

Thus, the model is very similar to the model of growth with externalities: aggregate production function exhibits increasing returns to scale even though the production function satisfies constant returns to scale from the final good producer's perspective.

- All in all this class of models, are very similar to the AK growth model. They all rely on linearity and convergence is immediate. The mechanics and the economics of growth however are very different and each is insightful about where growth comes from.

Fiscal Policies in Macroeconomic Models

We have mainly stayed away from government in our model but we cannot simply do so if we are try to write a macroeconomic model simply because of the level of involvement of government in the economy. There are various ways to think about the government and its policies:

1. Wasteful government spending: the most common way of modeling government is to consider its spending as wasteful and exogenous to our models. In this way, the only role for the government policy is to finance this wasteful spending via various kinds of taxes. This is the view that we will also use in most parts of what follows. The idea here is that government spending is determined through a complicated political process of voting, negotiations, legislation, etc. and this is perhaps exogenous to the macroeconomy.

2. Government as provider of public goods: Another view of government policy is that there are some goods who are inefficiently provided - goods that have large fixed cost of production and they are jointly used by some agents in the economy. This means government spending could potentially be endogenous as household's demand for the public good can depend on the state of the economy.

3. Government as the authority to remove market ineffiencies: In some of the growth models that we have discussed above, market outcomes are inefficient - for example in the Romer's model of growth with externalities or in the model with expanding varieties. As you have shown in your homework, government policy can come in these situations and potentially correct these distortions.

4. Government as provider of social insurance: There are various situations where it is hard to think that private markets can provide social insurance - insurance against being born in the wrong families, insurance against being hit by shocks to industries - think of an Uber driver, etc. Although there are private ways of providing social insurance: family, church, local community groups, etc. One observation that seems to be true is that government has replaced - although partially - some of these institutions. Under this view, government policy is there to provide assistance to the less fortunate or the retired. In this case, it is harder to take government spending as exogenous. This is an area where I have done most of my research but we wont be able to cover in this class.

To start thinking about the first part, we start from the simplest static model where there is homogenous set of agents who work and consume. Thus their preferences are given by $u(c, \ell)$. There is a firm that produces the final good using labor with a production function of the form $y = An$.

We consider various tax schemes:

1. Lump-sum taxes: Under lump-sum taxes, workers budget constarint is given by

$$c \leq w(1 - \ell) - T$$

2. Linear income tax: Under linear income taxes, workers budget constraint is given by

$$c \leq w(1-\ell)(1-\tau_l)$$

3. Consumption taxes: Under consumption taxes, workers budget constraint is given by

$$c(1+\tau_c) \leq w(1-\ell)$$

In general, given a tax schedule, we can modify our notion of equilibrium defined above to include government policy. In particular, consider the general case where all of the above are present. Then a Tax Distorted Competitive Equilibrium for this economy, or TDCE, given a set of policies, $\{T, \tau_l, \tau_c, G\}$ is defined as allocations $\left(\hat{c}, \hat{\ell}\right)$, prices $\hat{w}$, such that

i. Households maximize - given prices and policies:

$$\left(\hat{c}, \hat{\ell}\right) \in \arg\max u(c, \ell)$$

subject to

$$(1+\tau_c)c \leq \hat{w}(1-\tau_l)(1-\ell) - T$$

ii. Firms maximize - given prices:

$$1-\ell \in \arg\max An - \hat{w}n$$

iii. Government budget constraint is satsified

$$G = \hat{c}\tau_c + \hat{w}\hat{\ell}\tau_l + T$$

iv. Markets clear.

Analyzing TDCE in this environment is pretty straightforward. Given the tax policies, we can solve the consumer's problem and find the allocations that make government budget constraint holds.

Note that we could have also imposed taxes on the firm and distorting its hiring decision (Question: what kind of taxes would achieve that?). A seminal paper by Diamond and Mirrlees (1971) shows that with constant returns to scale there is no need to tax firms, i.e. no need to create inefficiency in production. This is really not relevant for this problem above as production efficiency is equivalent to taxation of households (why?). Yet their result applies in more general settings and can be used to show that there is no need to tax intermediate goods on the firm side - those that are used in production but not for consumption.

The key question that we would like to answer is what is a tax policy that is the best one. Note that due to the fact that government has some wasteful spending, the pareto optimal allocations defined before are not achievable any more. Yet still this question is an important one because we would like to have government pay for this wasteful spending in the least costly way possible. Now, let's compare scenarios:

1. Lump-sum taxes: Under lump-sum taxes, in any TDCE, $G = T$ - by government's budget balance. Thus, in any TDCE, the households must solve the following optimization

$$\max u\left(c, \ell\right)$$

subject to

$$c \leq A\left(1-\ell\right) - G$$

You can easily check that the solution to the above problem is equivalent to the following planning problem

$$\max u\left(c, \ell\right)$$

subject to

$$c + G = A\left(1-\ell\right)$$

This is a planning problem associated with a planner that simply takes G units of the final goods from the firm and gives it to the government. Thus as this shows, lump-sum taxes achieve the best thing possible.

2. Linear labor-income taxes: With such taxes, TDCE must satisfy

$$\max u\left(c, \ell\right)$$

subject to

$$c \leq \left(1-\tau_l\right) A\left(1-\ell\right)$$

and

$$G = \tau_l A\left(1-\ell\right)$$

where ℓ in the second equation is the solution of the optimization problem. To understand this, we need consider the behavior of government revenue for a given level of taxes τ_l. At $\tau_l = 0$, government revenue is 0 while at $\tau_l = 1$, it is also zero. Under fairly standard assumption on the utility function, government revenue is a hump-shape function. This is shown in the following figure: Either G is higher than G^* in which case no TDCE exists or G is lower than G^* in which case there are only two possible values of τ_l for which government's budget constraint holds. Thus, not every combination of τ_l and G is consistent with a TDCE. The curve above is the so-called Laffer curve which describes government's revenue as a function of tax rates.

The above analysis points to two main results. We cannot achieve the same outcome as under lump-sum taxes. This is because τ_l must be positive in a TDCE and that we must have

$$\frac{u_\ell\left(A\left(1-\ell\right)\left(1-\tau_l\right), \ell\right)}{u_c\left(A\left(1-\ell\right)\left(1-\tau_l\right), \ell\right)} = A\left(1-\tau_l\right)$$

which means that there are two distortions to labor supply, i.e., welfare is lower than the one achieved in the planning problem described above. Second it allows us to choose which tax rate is better for a given value of G. Note that although government revenue is hump-shaped in τ_l, welfare of the households is decreasing in τ_l (why?) and thus among the two value of τ_l that generates the same revenue for the government, the lower one provides a higher welfare for the households.

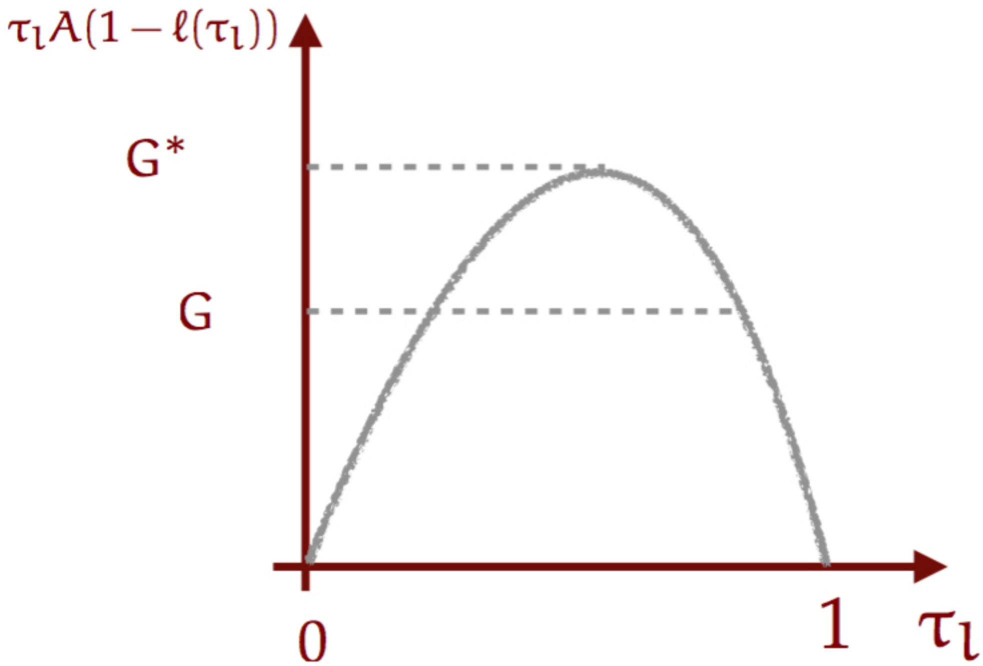

Figure 1: Government revenue as a function of the marginal income tax rate τ_l

3. Consumption taxes: The analysis of TDCE with consumption taxes follows closely the above analysis. You should try to show that best consumption tax and best labor income tax achieve the same level of welfare.

The above analysis of the best tax rate points toward an interesting principle: that government must always try to minimize distortions to efficiency while equating its revenue to its spending. We will try to formalize this in what follows.

In general, it is easy to characterize the solution of the best government policy when lump-sum taxes are available. Basically the problem would be to write down a planning problem where feasibility constraints are satisfied. It is, however, possible that lump-sum taxes are not available. Lump-sum taxes in their most general form are taxes that are a function of individual identities: I should pay a different tax than you because my name is different than you. This is not necessarily feasible by governments nor desirable - it is the basic form of discrimination which societies (the ones who are subject to it anyways!) typically dislike. Thus it is reasonable to ask what happens when we restrict attention to linear taxes where lump-sum taxes are not present. In other words, what is the best tax scheme when lump-sum taxes are unavailable.

There are two ways that I am aware of trying to solve the optimal taxation problem. The first is what was traditionally done in the literature - it goes back to Ramsey (1927)'s seminal paper. In this way of solving the problem we characterize the TDCE for a given set of tax functions and then we find the optimum by optimizing over possible taxes.

Let me illustrate this in a slightly different example than above. Suppose the economy is consisted of a homogenous group of individuals who earn labor income - valued at wage w - and want to consume a bundle of goods consisted of N different consumption goods. The government imposes consumption taxes as well as income taxes, at potentially different rates. Suppose that they are all produced using a technology that uses the labor whose productivity is normalized to 1. The government uses its tax revenue to finance its expenditure in each consumption good. Thus a TDCE is defined by allocations $\{c_1, \cdots, c_N, n_1, \cdots, n_N\}$, prices $\{w, p_i\}_{i=1,\cdots,N}$ such that given government policy $g_1, \cdots, g_N, \tau_1, \cdots, \tau_N$:

1. Households maximize, they solve

$$\max U(c_1, \cdots, c_N, \ell)$$

subject to

$$\sum_i p_i (1 + \tau_i) c_i \leq w (1 - \ell)$$

2. Firms maximize - they choose the amount of numeraire good to produce each consumption good i

$$n_i \in \arg\max p_i n - wn$$

3. Government's budget constraint holds:

$$\sum_i p_i \tau_i c_i = \sum_i p_i g_i$$

4. Markets clear:

$$c_i + g_i = n_i, \sum_i n_i = 1 - \ell$$

By walras law we can normalize wages to $w = 1$. Then optimality of the firm's decisions implies that

$$p_i = 1$$

Thus a TDCE can be characterized by allocations c_i that solve the following problem

$$(c_1, \cdots, c_N, \ell) \in \arg\max U(c_1, \cdots, c_N, \ell)$$

subject to

$$\sum_i (1 + \tau_i) c_i = 1 - \ell$$

and satisfies

$$\sum_i \tau_i c_i = \sum g_i$$

If we refer to the value of the above problem as $v(1 + \tau_1, \cdots, 1 + \tau_N)$, the government's optimal taxation problem is given by

$$\max v(1 + \tau_1, \cdots, 1 + \tau_N) \tag{P}$$

subject to

$$\sum_i \tau_i c_i (1 + \tau_1, \cdots, 1 + \tau_N) = \sum_i g_i$$

We can simply solve this problem using Lagrangian method but it is useful to do this using a tax pertrubation. In other words, suppose that we change the tax on good i by a small amount $\delta > 0$. Then the effect of this on budget and welfare are consisted of three effects:

- A mechanical effect on government budget: When we change τ_i, even absent any change in consumer behavior, the government revenue changes. This change is given by δc_i.
- A behavioral effect: A change in τ_i affects the consumption of all of the other goods. This change can be written in terms of uncompensated or Marshallian elasticities:

$$\delta \sum_j \tau_j \frac{\partial c_j}{\partial (1+\tau_i)} = \delta \sum_j \tau_j \varepsilon_{j,i} \frac{c_j}{1+\tau_i}$$

  where $\varepsilon_{j,i}$ is the uncompensated elasticity of demand for j with respect to change in price i. Intuitively, when the tax on i increases, assuming normality of goods, demand for i declines while the demand for other goods increases. In other words, $\varepsilon_{i,i} < 0$ and $\varepsilon_{j,i} > 0$ for reasonable utility functions.
- A welfare effect: Finally the change in the tax rate decreases the welfare of the households and this is given by

$$\delta \times v_{\tau_i} = -\delta \times U_\ell c_i$$

  where in the above we have used the Envelope condition associated with the value function v and the FOC associated with ℓ .

Now, at the optimum, these effects must sum up to zero and we should have

$$\lambda \left[c_i + \frac{1}{1+\tau_i} \sum_j \tau_j \varepsilon_{j,i} c_j \right] = U_\ell c_i$$

where λ is the multiplier associated with the government budget constraint in the optimal taxation above. We can write this as

$$\frac{1}{1+\tau_i} \sum_j \tau_j \frac{\varepsilon_{j,i} c_j}{c_i} = \lambda^{-1} U_\ell - 1$$

This formula is a complicated one yet it can be used to show various insights.

For example, suppose that $\varepsilon_{j,i} = 0$ for all $j \neq i$. An example of such preferences is $\sum_i \alpha_i \log c_i$ (show this!!), then the above formulas become

$$\frac{\tau_i}{1+\tau_i} = \frac{\lambda^{-1} U_\ell - 1}{\varepsilon_{i,i}}$$

Note that in this case $\varepsilon_{i,i}$ and $\lambda^{-1} u_\ell - 1$ are both negative. Thus goods that have a higher elasticity must be taxed at a lower rate. This is an interesting and intuitive principle that is commonly used in policy. Remember however that this principle is incomplete since it assumes that the cross price elasticities are always zero which is not a great assumption.

Now, let's go back to the case where we had Cobb-Douglass. In this case, the elasticities of all the goods are equal and thus taxes must be equal. This is a special case of the uniform commodity taxation result which shows that with homothetic utility functions, i.e., linear Engel curves, commodity taxes must be the same. This result was shown in some special cases by Atkinson and Stiglitz (1976) and later extended by Deaton (1979) and Deaton (1981).

Production Efficiency a la Diamond and Mirrlees (1971).

Now, let us try to extend the above analysis to an economy with production. In particular, suppose that consumer $i's$ problem can be written as

$$\max_{\mathbf{x}} u^i(\mathbf{x})$$

subject to

$$\mathbf{q} \cdot \mathbf{x} \leq 0$$

where in the above $\mathbf{x}$ is the vector of net-demand by the consumer and $\mathbf{q} \geq 0$ is the vector of after tax prices given by

$$\mathbf{q} = (q_1, \cdots, q_N) = ((1 + \tau_1^c)\, \hat{p}_1, \cdots, (1 + \tau_N^c)\, \hat{p}_N)$$

where τ_i^c is the ad-valorem tax on net use of good i and $\hat{p}_i$ is the price of good i in the tax-distorted equilibrium. Note that $\mathbf{x}$ is a fairly general notion of consumption. For example, it could include labor supply – as a negative number since it must enter the LHS of the budget constraint. Moreover, the government is unable to tax endowments since that would be equivalent to lump-sum taxes. Finally, all consumers face the same taxes and thus the same vector after tax prices. We refer to $\mathbf{q}$ as the vector of consumer prices. Let the indirect utility function associated with the above problem be given by $V^i(\mathbf{q})$ and the demand function for i be given by $\mathbf{x}^i(\mathbf{q})$.

On the production side, let production sets be defined by $G(\mathbf{y}) \leq \mathbf{0}$ where $G : \mathbb{R}^N \to \mathbb{R}^k$ where inputs enter it with a negative sign and outputs enter with a positive sign. Note that $\mathbf{y}$ has the same dimension as $\mathbf{x}$. We assume that $G(\cdot)$ is constant returns to scale. Let $\mathbf{p}$ be the vector of after tax prices for producers – producer prices – so that the problem of the firms becomes

$$\max_{\mathbf{y}} \mathbf{p} \cdot \mathbf{y}$$

subject to

$$G(\mathbf{y}) \leq \mathbf{0}.$$

Note that $G(\cdot)$ is only a function of goods that consumers care about. It can, however, be derived from more detailed production. For example: Suppose that there is one final consumption good that can be produced from an intermediate input and labor. The intermediate input in turn can be produced with labor. That is suppose that

$$y_1 = z^\alpha l_1^{1-\alpha}, z = l_2$$

Then, we can find $G(\cdot)$ by solving the following optimization problem

$$\max_{l_1 + l_2 = l} l_2^\alpha l_1^{1-\alpha}$$

where in the above l is total labor supplied and $l_2^\alpha l_1^{1-\alpha}$ is the total output of the final good. The total output in the above problem becomes $\alpha^\alpha (1-\alpha)^{1-\alpha} l$. Thus $G(\cdot)$ is given by

$$G(y_1, y_2) = y_1 + \alpha^\alpha (1-\alpha)^{1-\alpha} y_2$$

where in the above $y_2 = -l$. In this example, we can also think about what taxation of intermediate inputs do. Suppose that the government puts a tax on purchases of the intermediate input by the final good producer. Then the optimization problem faced by final good producers is given by

$$\max_{z,l_1} \hat{p}_1 z^\alpha l_1^{1-\alpha} - \hat{w} l_1 - \hat{p}_2 (1+t) z$$

Then

$$(1-\alpha) \hat{p}_1 z^\alpha l_1^{-\alpha} = \hat{w}$$
$$\alpha \hat{p}_1 z^{\alpha-1} l_1^{1-\alpha} = \hat{p}_2 (1+t)$$

From optimality condition for the producers of intermediate inputs, we have that $\hat{p}_2 = w$ and $z = l_2$. The above conditions then imply that

$$\frac{l_1}{l_2} = \frac{\alpha(1+t)}{1-\alpha} \to l_1 = \frac{\alpha(1+t)}{\alpha(1+t)+1-\alpha} l, l_2 = \frac{1-\alpha}{\alpha(1+t)+1-\alpha} l$$

where l is total labor. Therefore, total output of the final good is given by

$$\frac{(\alpha(1+t))^\alpha (1-\alpha)^\alpha}{\alpha(1+t)+1-\alpha} l < \alpha^\alpha (1-\alpha)^\alpha l$$

This means that by taxing intermediate inputs, the government can choose points in the interior the production set $G(\mathbf{y}) \leq 0$.

Finally, we assume that government must finance a vector $\mathbf{g} \in \mathbb{R}^N$. Therefore, an equilibrium is characterized by

$$\mathbf{y}(\mathbf{p}) = \sum_{i=1}^{I} \mathbf{x}^i(\mathbf{q}) + \mathbf{g}$$

Let $\mathbf{X}(\mathbf{q}) = \sum_{i=1}^{I} \mathbf{x}^i(\mathbf{q})$. Therefore, by choosing consumer and producer taxes, the government chooses a vector producer prices and consumer prices but it is constrained by the above market clearing condition. Note that in the above, we do not have to worry about profits since we have constant returns to scale.

Given this setup, the problem of optimal taxation is given by

$$\max_{\mathbf{q}} W\left(V^1(\mathbf{q}), \cdots, V^I(\mathbf{q})\right)$$

subject to

$$G(\mathbf{X}(\mathbf{q}) + \mathbf{g}) \leq \mathbf{0}$$

In the above, $W(\cdots)$ is the social welfare function that represents the objective of the government – the objective of the government in Diamond and Mirrlees (1971) actually is more general than this. We assume that W is increasing in each of its arguments.

Now the main result of Diamond and Mirrlees (1971) is that production is always efficient at the optimum. That is, we must have that $G(\mathbf{X}(\mathbf{q})+\mathbf{g}) = 0$. The idea of the proof is fairly simple. Suppose that $G(\mathbf{X}(\mathbf{q})+\mathbf{g}) < \mathbf{0}$. Then, any small change in $\mathbf{q}$ in any direction keeps the allocation within the production set. We can thus change consumer prices in a direction that increase consumers' utilities. There are some technical assumptions that must be made to show this formally but I refer you to reader their wonderful paper for the details.

This result is very important. It basically says that when we have a rich set of taxes, one for each good that consumers care about, then we should not tax intermediate inputs – production must be efficient. In words, we should leave production alone and just tax consumers! (Question: why is this implied?) Later, I will show how it implies uniform commodity taxation for homothetic utility function and can have implications for taxation of capital.

While the above is useful, it is hard to see that it should work for the type of environment that we are dealing: dynamic economies with infinitely many goods. An alternative approach that is now commonly used in the macro literature is to focus on allocations instead of policies. In particular, instead of solving a planning problem like (P) we would find the restriction that TDCE imposes on allocations and then find the best allocations among the set of allocations that can be outcome of a TDCE. We can then look at the FOCs to back out taxes. The benefit of this method is that it is mathematically very convenient but less intuitive than what we presented above.

To see this, consider the very first example of the economy with one final good and leisure and assume that lump-sum taxes are not present. Any allocation that is part of a TDCE must satisfy

$$u_c = \lambda(1+\tau_c), u_\ell = \lambda(1-\tau_l)w$$

Now if we multiply the budget constraint by λ and replace in the resulting relationship from above we have

$$\lambda c(1+\tau_c) = \lambda w(1-\ell)(1-\tau_l) \to cu_c = u_\ell(1-\ell)$$

Thus if an allocations is a result of a TDCE it must satisfy two conditions:

1. Implementability condition:

$$cu_c = u_\ell(1-\ell)$$

2. feasibility

$$c + G = A(1-\ell)$$

Interestingly enough, we can show that the reverse is actually true. In other words, if an allocation satisfies the above, then policy exists such that these allocations constitutes a TDCE. To see this, suppose that an allocation satisfies the above. We show that we can achieve this allocation in a TDCE with only labor income taxes. Let, tax rate be defined by

$$\tau_l = 1 - \frac{u_\ell}{Au_c}$$

Now consider the problem for the households given by

$$\max u(c,\ell)$$

subject to

$$c \le A(1-\ell)(1-\tau_l)$$

Since this is a strictly concave optimization problem, its solution must satisfy the first order condition as well as the budget constraint. By definition of τ_l , the allocation satisfies the FOC. In addition, it satisfies the budget constraint, which is guaranteed by the implementability condition. Now since consumer budget constraint is satisfied and allocation is feasible, the government budget constraint must hold as well. This proves that the desired allocation is part of a TDCE which completes the proof.

Now, the example with many goods is similar. We have

$$\begin{aligned} U_{c_i} &= \lambda\left(1+\tau_i\right)p_i \\ U_\ell &= \lambda w \end{aligned}$$

Then replacing in the budget constraint gives us

$$\sum_i U_{c_i}c_i = U_\ell\left(1-\ell\right)$$

Now you can show that if an allocation satisfies feasibility and the above then there is a TDCE which this allocation is a part of it.

Now we can write the Ramsey planning problem as

$$\max U\left(c_1,\cdots,c_N,\ell\right)$$

subject to

$$\begin{aligned} \sum_i U_{c_i}c_i &= U_\ell\left(1-\ell\right) \\ \sum_i c_i + g_i &= 1-\ell \end{aligned}$$

Suppose that $U\left(\mathbf{c},\ell\right)$ is separable between leisure and consumption goods; that is $\frac{\partial^2}{\partial\ell\partial c_i}U = 0$.Then taking FOC's we have

$$U_{c_i} - \lambda\sum_j U_{c_ic_j}c_j - \lambda U_{c_i} = \gamma$$

Now if we impose homotheticity, we know that

$$\frac{\sum_j U_{c_ic_j}c_j}{\sum_j U_{c_kc_j}c_j} = \frac{U_{c_i}}{U_{c_k}} \rightarrow \frac{\sum_j U_{c_ic_j}c_j}{U_{c_i}} = \frac{\sum_j U_{c_kc_j}c_j}{U_{c_k}}$$

Question: Can you prove this?

Diving the above first order condition by U_i and comparing it for i and k , we have

$$\begin{aligned} 1-\lambda\frac{\sum_j U_{c_ic_j}c_j}{U_{c_i}} - \lambda &= \frac{\gamma}{U_{c_i}} \\ 1-\lambda\frac{\sum_j U_{c_kc_j}c_j}{U_{c_k}} - \lambda &= \frac{\gamma}{U_{c_k}} \end{aligned}$$

Homotheticity of the utility function implies that the LHS of the above equalities are equal and thus $\frac{U_{c_i}}{U_{c_k}} = 1$. Since in any TDCE,

$$\frac{U_{c_i}}{U_{c_k}} = \frac{1+\tau_i}{1+\tau_k}$$

Therefore we have the result that these commodities must be taxed at the same rate. This result is an important result as it will provide the key intuition for our results in dynamic economies. Another way to derive this result is to actually use Diamond and Mirrlees (1971)'s result on production efficiency. To see this, since U is additively separable between $\mathbf{c}$ and ℓ it can be written as $U(\mathbf{c},\ell) = u(\mathbf{c}) - v(\ell)$. Now, since u is homothetic, a monotone transformation $f(\cdot)$ of it exists so that $f(u(\mathbf{c}))$ is homogeneous of degree 1. Now consider an economy that produces a final good x using a production function $f(u(\mathbf{c}))$. By construction, this production function is constant returns to scale. Suppose that the households in this economy only consume this final good and have a utility function $f^{-1}(x) - v(\ell)$. Then this economy would be equivalent to our economy above except that all the goods in $\mathbf{c}$ will be intermediate inputs to production. Using the Diamond and Mirlees's result, we see that production in this alternative economy must be undistorted and as a result the tax on all of these intermediate inputs should be identical. This proves the desired uniform commodity taxation result! We can also prove Deaton (1979)'s result – that if the utility function is homothetic over a subset of goods, then their taxes must be uniform using the same trick.

Now, we can turn to our dynamic economy as developed at the beginning of the class. We can similarly define TDCE under the assumption that there are no lump-sum taxes. Government can impose taxes on investment, consumption, capital income, labor income and also impose these at the firm level. Thus the households' problem in this TDCE is given by

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t, \ell_t)$$

subject to

$$\sum_{t=0}^{\infty} \left[p_{c,t}(1+\tau_{c,t})c_t + p_{x,t}(1+\tau_{x,t})x_t\right] \leq \sum_{t=0}^{\infty}\left[r_t(1-\tau_{k,t})k_t + w_t(1-\tau_{l,t})(1-\ell_t)\right]$$

$$k_{t+1} = k_t(1-\delta) + x_t$$

Similarly, firms' problems are given by

$$\max p_{c,t}\left(1-\tau_{c,t}^{c}\right)F^{i_c}\left(k_t^{i_c}, n_t^{i_c}\right) - w_t\left(1+\tau_{l,t}^{c}\right)n_t^{i_c} - r_t\left(1+\tau_{k,t}^{c}\right)n_t^{i_c}$$

$$\max p_{x,t}\left(1-\tau_{x,t}^{x}\right)F^{i_k}\left(k_t^{i_k}, n_t^{i_k}\right) - w_t\left(1+\tau_{l,t}^{x}\right)n_t^{i_k} - r_t\left(1+\tau_{k,t}^{x}\right)n_t^{i_k}$$

Government budget constraint must be satisfied

$$\sum_{t=0}^{\infty}\left[p_{c,t}\tau_{c,t}c_t + p_{x,t}\tau_{x,t}x_t + r_t\tau_{k,t}k_t + w_t\tau_{l,t}(1-\ell_t)\right] +$$

$$\sum_{t=0}^{\infty}\left[p_{c,t}\tau_{c,t}^{c}y_t^{c} + p_{x,t}\tau_{x,t}^{x}y_t^{x} + r_t\tau_{k,t}^{c}k_t^{c} + w_t\tau_{l,t}^{c}n_t^{c} + r_t\tau_{k,t}^{x}k_t^{k} + w_t\tau_{l,t}^{x}n_t^{k}\right] = \sum_{t=0}^{\infty} p_{c,t}g_t^{c} + p_{x,t}g_t^{x}$$

where g_t^c and g_t^x are government's purchases of consumption and investment goods. Note that in the background, government is actually borrowing or lending from the households. Now an allocation and prices given government policies is a TDCE if it solves the above problems, it satisfies the government budget constraint and it is feasible where feasibility is defined as usual.

It would be useful to analyze the equilibrium response of households to taxes in this model. To do so, let's assume that as usual consumption and investment goods are produced using the same production function. Furthermore, we can ignore taxes on the firm side since we can just push distortions to the consumer side and not allow for double taxation. Then, we can write the optimality condition associated with investment and labor supply in each period as

$$u_{c,t}\frac{1+\tau_{x,t}}{1+\tau_{c,t}} = \beta\frac{u_{c,t+1}}{1+\tau_{c,t+1}}\left[(1-\delta)\left(1+\tau_{x,t+1}\right)+F_{k,t+1}\left(1-\tau_{k,t+1}\right)\right] \tag{8}$$

$$\frac{u_{\ell,t}}{u_{c,t}} = F_{l,t}\frac{1-\tau_{\ell,t}}{1+\tau_{c,t}} \tag{9}$$

As we see, compared to an economy with lump-sum taxes where regular Euler equations hold, in the above, taxes do affect saving and work decision. However, there are some tax schedules that do not affect each decision. For example, suppose that the only tax is a constant tax on consumption. Then, the investment decision is undistorted in that on the margin the Euler equation looks like that in the undistorted model. Note that with consumption taxes, the labor supply margin is distorted since consumption tax acts similar to a labor income tax. On the other hand, if we have positive capital income taxes, $\tau_{k,t} = \tau_k > 0$, then the investment decision is distorted. If in addition, we allow for a subsidy on investment – and ignore whether government budget constraint holds or not – and the size of this subsidy is $-\tau_k$, then inestment becomes undistorted. As Abel (2007) shows, this is equivalent to a corporate income tax where firms are allowed to deduct their investment expenses.

Now turning to characterization of TDCE, as before, we can show that a TDCE is equivalent to an implementability condition and feasibility. This implementability condition is given by

$$\sum_{t=0}^{\infty}\beta^t\left[u_{ct}c_t - u_{\ell t}\left(1-\ell_t\right)\right] = \frac{u_{c0}}{1+\tau_{c0}}\left[(1-\delta)\left(1+\tau_{x,0}\right)+F_{k,0}\left(1-\tau_{k,0}\right)\right]k_0$$

So now, if we normalize price of consumption at period 0 to 1, then we have the following Ramsey planning problem

$$\max\sum_{t=0}^{\infty}\beta^t u\left(c_t,\ell_t\right)$$

subject to

$$c_t + g_t + k_{t+1} = k_t\left(1-\delta\right) + F\left(k_t, 1-\ell_t\right)$$

$$\sum_{t=0}^{\infty}\beta^t\left[u_{ct}c_t - u_{\ell t}\left(1-\ell_t\right)\right] = \frac{u_{c0}}{1+\tau_{c0}}\left[(1-\delta)\left(1+\tau_{x,0}\right)+F_{k,0}\left(1-\tau_{k,0}\right)\right]k_0$$

$$k_0 : \text{given}$$

Now, consider this problem mathematically. We can conjecture that the implementability constraint is slack and solve for optimal allocation – it is as if we have lump-sum taxes! Once we do that we can just choose $\tau_{c,0}$ with $\tau_{x,0} = \tau_{k,0} = 0$ so that the implementability constraint is satisfied. What are taxes doing in the background. If we tax consumption at date 0, and want to implement solution with lump-sum taxes, we must set a constant consumption tax in all periods and subsidize labor supply at an equal rate. (Show why this actually works!) Now, what does this mean? It points to a special feature of the dynamic model which is that time-0 capital is a factor of production that is not produced. Therefore, it is possible to tax it without really causing distortions. Because of this, we have to make some restrictions on time-0 taxes. The easiest would be to assume that they are zero.

Another feature of this problem that is related to time-0 capital is worth discussing. Note that as we mentioned time-0 capital is special in that taxing it will have no behavioral effect. This is contrary to all future capitals where their taxation leads to under-saving, i.e., a behavioral response by the households. This is an important difference between this problem and the standard growth models that we have seen in the past. It creates a time-inconsistency issue. The government at time 0 would like to set some value for capital taxes at $t = 1$ but at $t = 1$, a government that makes decision would like to deviate, because all of the investment done in the past is sunk. This time inconsistency problem is an important issue but it is beyond the scope of our class. You can read some more about it in the paper by Chari and Kehoe (1990).

We can take FOC of this problem and write

$$\beta^t u_{ct} - \lambda \beta^t \left[u_{ct} + u_{cct} c_t - u_{\ell c,t} \left(1 - \ell_t\right)\right] = \mu_t$$
$$\mu_{t+1} \left(1 - \delta + F_{k,t+1}\right) = \mu_t$$

If we assume that as time grows, allocations converge to a non-zero steady state, we must have that

$$\frac{\mu_{t+1}}{\mu_t} = \beta \frac{u_{ct+1} - \lambda \left[u_{ct+1} + u_{cct+1} c_{t+1} - u_{\ell c,t+1} \left(1 - \ell_{t+1}\right)\right]}{u_{ct} - \lambda \left[u_{ct} + u_{cct} c_t - u_{\ell c,t} \left(1 - \ell_t\right)\right]} \to \beta$$

as $t \to \infty$. This combined with the FOC above implies that

$$\lim_{t \to \infty} 1 - \delta + F_{k,t+1} = \frac{1}{\beta}$$

Note that in a TDCE, the Euler equation for the household is given by

$$1 - \delta + F_{k,t+1} \left(1 - \tau_{k,t}\right) = \frac{u_{c,t}}{\beta u_{c,t+1}}$$

These two equations combined with the fact that allocations converge to steady state implies that

$$\lim_{t \to \infty} \tau_{k,t} = 0$$

This is the celebrated zero capital tax in the long-run result as shown (and not necessarily in this environment) by Chamley (1986) and Judd (1985). It is fairly robust (there are some exceptions and is closely related to the uniform commodity taxation result that we discussed before. To see this, suppose that $u(c) = \frac{c^{1-\sigma}}{1-\sigma} + v(\ell)$; a semi-homothetic utility function. Then we have

$$\beta^t c_t^{-\sigma} - \lambda \beta^t \left[\left(1 - \sigma\right) c_t^{-\sigma}\right] = \mu_t$$

Therefore

$$1-\delta+F_{k,t+1}=\frac{\mu_{t+1}}{\mu_t}=\frac{\beta^{t+1}c_{t+1}^{-\sigma}\left(1-\lambda\left(1-\sigma\right)\right)}{\beta^t c_t^{-\sigma}\left(1-\lambda\left(1-\sigma\right)\right)}=\beta\frac{c_{t+1}^{-\sigma}}{c_t^{-\sigma}}$$

The above relationship implies that the undistorted Euler equation must be satisfied and therefore $\tau_{k,t+1}=0, \forall t\geq 1$. This is important since it says that capital taxes are always zero - well from period 2 onwards. Other versions of this result exists; for example Atkinson and Stiglitz (1976) show that even with non-linear labor income taxes and in a two period model saving taxes must be zero - with homothetic utility functions.

Introducing Shocks to Our Models

A large part of the macro literature, too large if you ask me(!), is concerned with business cycle fluctuations. Short term movements in macro variable around the trend. So far, what we have talked about has been mainly focused on long-term movements in these variables, so we'd like to focus on the wiggles around the trend. The key question is how to create fluctuations in macro variables in our model.

One approach that does not generate a lot of interest because it is not necessarily very easy to do is to write down models where the dynamics are inherently cyclical. For example, if in one period output is above its trend then the next period - through some economic forces - it will be below and so on. Although people have written models like this, including yours truly, but the problem with this approach is that unless the models have multiple equilibria, it will be easy to predict the behavior of the economy, if these cyclical fluctuations are deterministic - as is the case in the example I described. Now, as Paul Samuelson has famously said: " Wall Street has predicted nine out of the last five recessions!" In other words, it does not look like we are great at predicting recessions let alone be able to with absolute certainty explain what will happen to macro variables in the next quarter or year. This suggests that in order to study business cycles, we better study models where there are shocks.

Now, let's consider our baseline growth model. What can we shock? Pretty much anything we like! We can shock TFP, saying that unpredictable innovations can lead to an increase in TFP while unpredictable increases price of oil can lead to a decline in TFP (*why?*). We can shock government spending, saying that political interactions could be unpredictable and lead to changes in G. We can also shock taxes using the same reasoning. We can shock preferences by saying that people's tastes for consumption changes in an unpredictable way (there are also more complicated stories for this).

Now, whatever you shock, there will be an exogenous state to the model which we represent by $s_t\in\mathcal{S}$ - it is simplest to assume that $\mathcal{S}=\{\overline{s}\left(1\right),\cdots,\overline{s}\left(N\right)\}$ but we can also use continuous states; for now we stick to discrete states to make the problem easier technically. Here, s_t represents part of the state of the economy that is exogenously being shocked and is governed by some stochastic process. For example, s_t could include any of the shocks that we discussed above. It is useful to define $s^t=\left(s_0,s_1,\cdots,s_t\right)$ as the history of the shocks that the economy has experienced. This is probably a better description of the exogenous state of the economy. Now a stochastic process basically imposes a probability distribution over future outcomes. In particular, we can write

$$\Pr\left(s_{t+1}|s^t\right)=\pi\left(s_{t+1}|s^t\right)$$

Here $\pi\left(s_{t+1}|s^t\right)$ is the conditional distribution of s_{t+1} given what has happened to the economy in the past, s^t. Using this function, we can construct the entire distribution of future events given what has happened to the economy in the past. A special case of this is a first order Markov process where $\pi\left(s_{t+1}|s^t\right) = \pi\left(s_{t+1}|s_t\right)$, i.e., the conditional distribution of tomorrow's state depends on the state today. With a little bit of abuse of notation, we let $\pi\left(s^t\right)$ be the unconditional probability of the history s^t.

A simplifying assumption that we will make is that everyone in the economy knows the probability distribution $\pi\left(s_{t+1}|s^t\right)$. In other words, everyone knows exactly what stochastic process governs the economy and they all agree on it. This is the so-called rational expectations assumption that greatly simplifies our analysis. Obviously this assumption is incorrect and people do not have the same beliefs. Nevertheless, it makes life simple and allows to solve an otherwise complicated model. More recently, some people have relaxed this assumptions, for example papers by Angeletos and La'O (2013) or Huo and Takayama (2015).

Given the above setup, allocations are functions of histories. In the context of our model, allocations are given by

$$\left\{c_t\left(s^t\right), x_t\left(s^t\right), k_{t+1}\left(s^t\right), \ell_t\left(s^t\right)\right\}_{t\geq 0, s^t\in\mathcal{S}^{t+1}}$$

Note that capital at $t+1$ is only a function of s^t. This is because investment is really done at time t and thus depends on s^t.

We can use the above allocations to define payoffs which is given by

$$\sum_{t=0}^{\infty}\sum_{s^t\in\mathcal{S}^t}\beta^t\pi\left(s^t\right)u\left(c_t\left(s^t\right), \ell_t\left(s^t\right)\right)$$

What remains to be explained is how trading is done. One way to do the trading is to assume as before that all the trading occurs at time-0 before any state is realized. Under this interpretation, the assumption is that people trade futures contracts with each other to deliver consumption goods, labor services, rent out capital and investment goods. With time-0 trading, there is a single budget constraint given by

$$\sum_{t=0}^{\infty}\sum_{s^t}p_{c,t}\left(s^t\right)\left[c_t\left(s^t\right)+x_t\left(s^t\right)\right] \leq \sum_{t=0}^{\infty}\sum_{s^t}\left[r_t\left(s^t\right)k_t\left(s^{t-1}\right)+w_t\left(s^t\right)\left(1-\ell_t\left(s^t\right)\right)\right]$$

Notice that there are no probabilities in the above budget constraint. (*why?*)

We can also think about sequential trading in this model. To do so, we need to use certain securities. In particular, suppose that in each period t and state s^t, there exists a set of securities called Arrow securities that pay a unit of consumption in a particular state of the world in $t+1$. In other words, suppose that $a_{t+1}\left(s_{t+1}, s^t\right)$ is the number of units of the arrow security purchased at s^t that pays a unit of consumption if the state of the economy tomorrow is s_{t+1}. Then, we have the following budget constraint

$$c_t\left(s^t\right)+x_t\left(s^t\right)+\sum_{s_{t+1}}q_{t+1}\left(s_{t+1}, s^t\right)a_{t+1}\left(s_{t+1}, s^t\right) \leq \hat{r}_t\left(s^t\right)k_t\left(s^{t-1}\right)+\hat{w}_t\left(s^t\right)\left(1-\ell_t\left(s^t\right)\right)+a_t\left(s_t, s^{t-1}\right)$$

where in the above $q_{t+1}\left(s_{t+1}, s^t\right)$ is the price of the arrow security. While this market structure seems unrealistic at first glance, it is possible that other market structures are equivalent to this.

In particular, suppose that $s_{t+1} \in \{s_l, s_h\}$ and suppose the securities being traded are: 1. a risk-free bond that pays one unit of consumption good at $t+1$ in each state, 2. a short-lived stock that pays s_{t+1}. Consider a portfolio that is long the bond by $\frac{s_h}{s_h - s_l}$ and short the stock by $\frac{1}{s_h - s_l}$ units. The payoff this portfolio at $t+1$ is

$$\frac{s_h}{s_h - s_l} - \frac{s_{t+1}}{s_h - s_l} = \begin{cases} 1 & s_{t+1} = s_l \\ 0 & s_{t+1} = s_h \end{cases}$$

In other words, it replicates an Arrow security that pays off when $s_{t+1} = s_l$. Similarly, we can have portfolio that replicates an arrow security that pays when $s_{t+1} = s_h$. In other words, whatever consumption plan that is achieved by Arrow securities can be achieved by these two assets. This simple example illustrates that if we have enough securities, we can basically assume that we have all the arrow securities. This a spanning result; the two securities are said to fully span the payoff space. There are various theorem related to spanning: an interesting one is due to Duffie and Huang (1985). They show that even if the state of the economy is represented by N brownian motions, then having N stocks and one bond is enough to span the payoff space. As usual, we have to impose a transversality or no-ponzi-scheme condition:

$$Q_{t+1}\left(s^{t+1}\right) a_{t+1}\left(s_{t+1}, s^t\right) \to_{\text{a.s.}} 0$$

where

$$Q_{t+1}\left(s^{t+1}\right) = q_0\left(s_0\right) q_1\left(s_1, s^0\right) q_2\left(s_2, s^1\right) \cdots q_{t+1}\left(s_{t+1}, s^t\right)$$

Note that market clearing condition, besides the usual feasibility is given by

$$\sum_{i \in I} a^i_{t+1}\left(s_{t+1}, s^t\right) = 0$$

where I is the set of households. In other words, these financial assets are in zero net supply. This implies that in a setting with a representative household, there will be no trading equilibrium. We can still use optimality to price the assets.

How do we solve for a competitive equilibrium? The easiest one is to use efficiency. We can then write down a planning problem whose solution will be a competitive equilibrium:

$$\max_{c_t, \ell_t, x_t, k_{t+1}} \sum_{t=0}^{\infty} \sum_{s^t \in \mathcal{S}^t} \beta^t \pi\left(s^t\right) u\left(c_t\left(s^t\right), \ell_t\left(s^t\right)\right)$$

subject to

$$\begin{aligned} c_t\left(s^t\right) + x_t\left(s^t\right) + g\left(s^t\right) &= A_t\left(s^t\right) k_t\left(s^{t-1}\right)^{\alpha}\left(1 - \ell_t\left(s^t\right)\right)^{1-\alpha} \\ k_{t+1}\left(s^t\right) &= k_t\left(s^{t-1}\right)(1-\delta) + x_t\left(s^t\right) \\ k_0 &: \text{ given} \\ s_0 &: \text{ given} \end{aligned}$$

For simplicity, let's assume that s_t follows a first order Markov process. We can use our usual dynamic programming techniques to show that solving this sequence problem is equivalent to solving the following function equation

$$v(k, s) = \max u(c, \ell) + \beta \sum_{s'} \pi\left(s'|s\right) v\left(k', s'\right)$$

subject to

$$c + k' + g(s) = A(s) k^{\alpha} (1-\ell)^{1-\alpha} + (1-\delta) k$$

I will leave it to you to look up the necessary conditions in SLP to ensure that the above has a unique solution and satisfies various regularities: concave, increasing, etc.

A special case of this problem is one where $u(c,\ell) = \log c$, $\delta = 1$, $g(s) = 0$ and s is i.i.d. In other words, the only source of shock is i.i.d. shocks to TFP. We can then write the above in a simple fashion

$$\hat{v}(A(s) k^{\alpha}) = \max \log c + \beta \sum_{s'} \hat{v}\left(A(s') (k')^{\alpha}\right)$$

subject to

$$c + k' = A(s) k^{\alpha}$$

The reason we can do this is that with the i.i.d. assumption, A and k^{α} always appear together and as a result Ak^{α} is the sufficient statistic to know what will happen in the future. It is straightforward to guess and verify that

$$v(Ak^{\alpha}) = B_1 \log Ak^{\alpha} + B_2$$

for some $A_1 > 0$ and A_2. We can also show $k' = \beta A k^{\alpha}$. If we write this in terms of time, we have

$$k_{t+1} = \beta A_t k_t^{\alpha} \rightarrow \log k_{t+1} = \log \beta + \log A_t + \alpha \log k_t$$

Interestingly, $\log k_t$ is an AR(1) process with auto-regression coefficient α. In what follows, we will try to analyze the behavior of this dynamic stochastic system and other similar ones.

Stationarity

We are interested in analyzing the behavior of these dynamic systems when there are shocks involved. To do so, it is useful to define the concept of stationarity. A stochastic process, $\{x_t\}_{t=0}^{\infty}$ is said to be stationary if for any sequence of integers, $i_1 < i_2 < \cdots < i_n$, the joint distribution of $(x_{t+i_1}, \cdots, x_{t+i_n})$ is independent of t. In words, stationary processes are kind of independent of time. One way to think about this is to think of as an extension of the steady state definition or BGP to environments with shocks. Examples of this are:

1. $x_t = \varepsilon_t$ where $\varepsilon_t \sim G(\varepsilon_t)$ is identically and independently distributed.

2. $x_{t+1} = \rho x_t + \varepsilon_{t+1}$ where ε_t is identically and independently distributed and $|\rho| < 1$. This is an AR(1) process. Note that $|\rho| < 1$ is required for stationarity. To see this, suppose that $\rho = 1$, then

$$Var(x_t) = Var\left(x_0 + \sum_{s=1}^{t} \varepsilon_s\right) = tVar(\varepsilon)$$

Thus the variance of x_t depends on t which is at odds with the definition of a stationary process.

3. $x_{t+1} = \sum_{i=N-1}^{0} \rho_i x_{t-i} + \varepsilon_{t+1}$. This is an AR(N) process and is stationary if $\sum \rho_i^2 < 1$.

As mentioned, this concept of stationarity helps us characterize the behavior of our dynamic systems. Another useful concept that goes hand in hand with stationarity is ergodicity. We illustrate this through an example. Consider the solution of the dynamic programming above and assume that $A_t \in \{A_L, A_H\}$ with $A_L < A_H$. Let k_H and k_L be given by

$$k_H = \beta A_H k_H^{\alpha}, k_H = \beta A_L k_L^{\alpha}$$

These are level of capital that if the state was given by A_H or A_L forever would be the steady state of the model. It can be shown that for every long enough sequence of shocks, $k_t \in [k_L, k_H]$. In other words, if we let this economy run, in the long-run the values of capital - although stochastic - belong to this interval. In other words, this interval is the ergodic set of capital. Formally, consider a Bellman equation in stochastic form

$$v(x,s) = \max_{x' \in \Gamma(x,s)} F(x, x', s) + \beta \int v(x', s') \, dF(s'|s)$$

where $x \in \mathcal{X}$ and $s \in \mathcal{S}$. Let $x'(x,s)$ be its associated policy function. Then, the set $\mathcal{V} \subset \mathcal{X} \times \mathcal{S}$ is the ergodic set associated with $x'(x,s)$ if and only if

$$x'(\mathcal{V}) = \mathcal{V}$$

In the context of the above example, the ergodic set for the system is given by

$$[k_L, k_H] \times \{A_L, A_H\}$$

This is because A_t is i.i.d. Similarly, we can define the distribution induced by the stochastic process associated with s on the set $\mathcal{V}$. In particular, a stationary distribution for this dynamic system, $\mu \in \Delta(\mathcal{V})$ [2], satisfies

$$\mu(A) = \int \mathbf{1}\left[(x'(x,s), s') \in A\right] dF(s'|s) \, d\mu \tag{10}$$

There are various results in SLP and the paper by Hopenhayn and Prescott (1992) that show that under certain conditions, the dynamic stochastic system described above converges in distribution to μ. Typically, we assume that this convergence has occurred and calculate moments using μ.

Computations.

Now, we can solve this problem as in the case of the model with shocks via value function iteration. As it turns out, VFI is not very fast in solving these problems. Over the years people have come up with alternative methods: policy function iteration, finite element method etc. A more common method for solving this problem is via log-linearization. In this method, we write the Euler equations and feasibility for the above problem:

$$\begin{aligned} u_{c,t} &= \beta \mathbb{E}_t \left[\left(1 - \delta + \alpha A_{t+1} k_{t+1}^{\alpha-1} (1 - \ell_{t+1})^{1-\alpha}\right) u_{c,t+1} \right] \\ \frac{u_{\ell,t}}{u_{c,t}} &= (1-\alpha) A_t k_t^{\alpha} (1 - \ell_t)^{-\alpha} \\ c_t + k_{t+1} &= k_t (1 - \delta) + A_t k_t^{\alpha} (1 - \ell_t)^{1-\alpha} \end{aligned}$$

We then assume that the fluctuations are kind of small and close to the non-stochastic steady state of the model. We then approximate the above equations by assuming that all of them linear in terms of the log deviations from steady state. In other words, for each variable x_t and its steady state value x_{ss}, we let $\hat{x}_t = \log(x_t / x_{ss})$. We then write the above equation as linear functions of the hat variables. Then the above becomes a dynamic linear system for which there are by now

[2] For any set S, $\Delta(S)$ is the set of all probability measure over S.

standard methods that can help us solve them. The most straightforward is the method developed by Blanchard and Kahn (1980).

The last part of the computations typically involve the calculation of the stationary distribution which will be used in computing moments. To calculate the stationary distribution, we typically discretize the state space $\mathcal{X} = \{x_1, \cdots, x_m\}, \mathcal{S} = \{s_1, \cdots, s_n\}$. Under this discretization, the stationary distribution becomes a vector $\mu_{i,j}$ which is the probability of (x_i, s_j). Then, from the definition of the stationary distribution (10), we must have

$$\mu_{i,j} = \sum_{i',j'} \mathbf{1}\left[x'\left(x_{i'}, s_{j'}\right) = x_i\right] \pi\left(s_j | s'_j\right) \mu_{i',j'}$$

where $\pi^s(\cdot|\cdot)$ is the discretization of $F(\cdot|\cdot)$. The above is a linear system of equations which can be simply solved. We can then use this distribution to calculate moments of the function.

Asset Pricing

The framework that we have developed above allows us to price any asset. In particular, let $q\left(s_{t+1}, s^t\right)$ be the price of arrow securities. We can use these prices to determine the price of any asset using no-arbitrage conditions. Some examples:

- A real bond: Consider a real bond that pays one unit of the consumption good in all states in the future. No arbitrage condition becomes

$$Q_t^B\left(s^t\right) = \sum_{s_{t+1}} q_{t+1}\left(s_{t+1}, s^t\right)$$

- Stock: Consider a stock that has an associated dividend process $d_t\left(s^t\right)$. Then no arbitrage implies that

$$Q_t^S\left(s^t\right) = \sum_{s_{t+1}} q_{t+1}\left(s_{t+1}, s^t\right)\left[d_{t+1}\left(s^{t+1}\right) + Q_{t+1}^s\left(s^{t+1}\right)\right]$$

One can iterate on this equation and show that

$$Q_t^S\left(s^t\right) = \sum_{\tau=t+1}^{\infty} \prod_{s^\tau \succcurlyeq s^t} q_\tau\left(s^\tau\right) d_\tau\left(s^\tau\right) + \lim_{\tau\to\infty} q_{t+1}\left(s^{t+1}\right)\cdots q_\tau\left(s^\tau\right) Q_\tau^S\left(s^\tau\right)$$

The first term in the above equation is the fundamental value of the asset while the second term is sometimes referred to as a bubble; i.e., any deviation for the value of the asset from its fundamental value. Note that if the prices of arrow securities are on average strictly less than 1 and stock prices are stationary, then there cannot be any bubble. For more analysis on this, you can read the very nice paper by Santos and Woodford (1997) where they show that bubbles cannot arise under complete markets unless $d_\tau\left(s^\tau\right) = 0$ almost surely.

- Call options with strike price $\bar{q}$:

$$Q_t^{CO}\left(s^t\right) = \sum_{s_{t+1}} q_{t+1}\left(s_{t+1}, s^t\right) \max\left\{0, Q_{t+1}^S\left(s^{t+1}\right) - \bar{q}\right\}$$

Another popular way of writing the asset pricing equations above for an arbitraty asset is

$$Q_t = \mathbb{E}_t \left[m_{t+1} \left(d_{t+1} + Q_{t+1} \right) \right]$$

where $m_{t+1} = \beta \frac{u_{c,t+1}}{u_{c,t}}$ is sometimes called the stochastic discount factor or the pricing kernel. It can be shown in general that asset price processes that satisfy no arbitrage must satisfy the above equation for some pricing kernel.

Equity Premium Puzzle

The above analysis illustrates how to price assets using the SDF or what is sometimes called as state prices. Note that in our particular model, Euler equation for the households implies that state prices satisfy

$$q_{t+1} \left(s_{t+1}, s^t \right) = \beta \pi \left(s_{t+1} | s^t \right) \frac{u_c \left(s^{t+1} \right)}{u_c \left(s^t \right)}$$

Following Lucas Jr (1978), a large literature in Finance tries to examine whether our macro models can be consistent with the observed behavior of asset prices. Note that in our model, if we assume that $u(c, \ell) = \frac{c^{1-\gamma}}{1-\gamma} + v(\ell)$, then for any asset we must have

$$1 = \sum_{s_{t+1}} \beta \pi \left(s_{t+1} | s^t \right) \frac{c_{t+1} \left(s^{t+1} \right)^{-\gamma}}{c_t \left(s^t \right)^{-\gamma}} R_{t+1} \left(s^{t+1} \right)$$

where R_{t+1} is the return on the asset; in case of a stock it is $\frac{Q_{t+1} + d_{t+1}}{Q_t}$ while in case of a bond, its simply its return. We can write this in short form

$$1 = \mathbb{E}_t \left[\beta \frac{c_{t+1}^{-\gamma}}{c_t^{-\gamma}} R_{t+1} \right] \tag{11}$$

This is basically an Euler equation. The object $m_{t+1} = \beta \frac{c_{t+1}^{-\gamma}}{c_t^{-\gamma}}$ is sometimes referred to as pricing kernel or stochastic discount factor. In general, it can be shown that arbitrage free asset prices, must have a stochastic discount factor m_{t+1} where $\mathbb{E}_t \left[m_{t+1} R_{t+1} \right] = 1$. Now the system of equations in (11) is something that we can test using data on asset returns and data on aggregate consumption.

One simple exercise would be to allow consider the equations for bonds and the aggregate stock market. Suppose that we assume the following econometric specification for the stochastic processes driving consumption growth, return on total stock market, and return on bonds.

$$\begin{aligned} R_{t+1}^s &= \overline{R}^s e^{\varepsilon_{t+1}^s - \frac{1}{2}\sigma_s^2} \\ R_{t+1}^b &= \overline{R}^b e^{\varepsilon_{t+1}^b - \frac{1}{2}\sigma_b^2} \\ \frac{c_{t+1}}{c_t} &= \overline{c}_\Delta e^{\varepsilon_{t+1}^c - \frac{1}{2}\sigma_c^2} \end{aligned}$$

where

$$\begin{pmatrix} \varepsilon_t^s \\ \varepsilon_t^b \\ \varepsilon_t^c \end{pmatrix} \sim \mathcal{N}(0, \Sigma)$$

where

$$\sum = \begin{bmatrix} \sigma_s^2 & \sigma_{sb} & \sigma_{sc} \\ \sigma_{sb} & \sigma_b^2 & \sigma_{bc} \\ \sigma_{sc} & \sigma_{bc} & \sigma_c^2 \end{bmatrix}$$

We have

$$\mathbb{E}_t R_{t+1}^s = \overline{R}^s, \mathbb{E}_t R_{t+1}^b = \overline{R}^b$$

Note that we assume the return on bonds is risky since treasuries are nominal and therefore carry inflation risk while the market for TIPS (Treasury Inflation Protected Securities whose returns are indexed to inflation) is not very deep in the U.S. In any case, inflation risk is very small relative to the risk in stocks. Note that we have assumed that the returns as well as consumption growth are is i.i.d. processes.

The equation (11) becomes

$$1 = \mathbb{E}_t \left[\beta \overline{c}_\Delta^{-\gamma} e^{-\gamma\left(\varepsilon_{t+1}^c - \frac{1}{2}\sigma_c^2\right)} \overline{R}^s e^{\varepsilon_{t+1}^s - \frac{1}{2}\sigma_s^2} \right]$$
$$1 = \beta \overline{c}_\Delta^{-\gamma} \overline{R}^s e^{\frac{\gamma}{2}\sigma_c^2 - \frac{1}{2}\sigma_s^2} \mathbb{E}_t \left[e^{\varepsilon_{t+1}^s - \gamma\varepsilon_{t+1}^c} \right]$$

The random variable $\varepsilon_{t+1}^s - \gamma\varepsilon_{t+1}^c$ is a normal random variable with mean zero and variance

$$\sigma_s^2 + \gamma^2\sigma_c^2 - 2\gamma cov\left(\varepsilon^s, \varepsilon^c\right)$$

Therefore

$$\mathbb{E}_t \left[e^{\varepsilon_{t+1}^s - \gamma\varepsilon_{t+1}^c} \right] = e^{\frac{1}{2}\left(\sigma_s^2 + \gamma^2\sigma_c^2 - 2\gamma cov\left(\varepsilon^s, \varepsilon^c\right)\right)}$$

and we have

$$1 = \beta \overline{c}_\Delta^{-\gamma} \overline{R}^s e^{\frac{\gamma}{2}\sigma_c^2 - \frac{1}{2}\sigma_s^2} e^{\frac{1}{2}\left(\sigma_s^2 + \gamma^2\sigma_c^2 - 2\gamma cov\left(\varepsilon^s, \varepsilon^c\right)\right)}$$

Taking $\log$

$$0 = \log\beta - \gamma\log\overline{c}_\Delta + \log\overline{R}^s + \frac{1}{2}\left(\left(\gamma^2 + \gamma\right)\sigma_c^2 - 2\gamma\sigma_{sc}\right)$$

We can similarly write

$$0 = \log\beta - \gamma\log\overline{c}_\Delta + \log\overline{R}^b + \frac{1}{2}\left(\left(\gamma^2 + \gamma\right)\sigma_c^2 - 2\gamma\sigma_{bc}\right)$$

If we subtract the top from the bottom, we get

$$\log\overline{R}^s - \log\overline{R}^b = \gamma\left(\sigma_{sc} - \sigma_{bc}\right)$$

If we write the above in terms of net returns and use an approximation, we have

$$\overline{r}^s - \overline{r}^b = \gamma\left(\sigma_{sc} - \sigma_{bc}\right)$$

The LHS of the above relationship is the spread between average return on the stock market and average return on bonds. The RHS is the risk-aversion parameter and the difference between covariance of consumption growth and stock returns and covariance of consumption growth and bond returns. In the data, average return on stocks is %7, average return on bonds is %1, while the covariance between stocks and consumption growth is %0.219 and covariance of consumption growth and bonds is %-.0193. Therefore, in order for the above to hold, we must have

$$\gamma = \frac{.06}{.00219 + .000193} \approx 25.18$$

A value of 25 for risk-aversion, means that if an individual has on average a consumption of \$50000 and has a 50-50 chance of winning or loosing \$5000, then she is willing to give up around \$5015 in order to avoid this risk. This seems implausibly high. In fact most economists would think that a value above 10 is high. This implies a failure for the model. In other words, the holders of the stock have a very smooth consumption which does not vary a lot with the return on stocks, yet they are demanding a very high premium in order to hold stocks. The only way this can work is to have the holders of the stock to be extremely risk-averse which does not seem very plausible. This is the equity premium puzzle as illustrated by Mehra and Prescott (1985).

Here are a few resolutions for the EPP:

1. Changes in the utility function: Perhaps the utility function and their implied SDF is misspecified.

    (a) Habit formation a la Campbell and Cochrane (1999). Campbell and Cochrane (1999) assume that an accumulated measure of consumption affects the SDF. They show that this can generate fluctuations in the SDF even for low values of risk-aversion; enough to explain the EPP.

    (b) Recursive preferences: An important feature of the preferences considered above is that the intertemporal elasticity of substitution, or households' preferences for smoothness of consumption, is tied to the parameter of risk-aversion. In fact, IES is $\frac{1}{\gamma}$ while risk-aversion is γ. It is perhaps possible that if we adjust preferences to separate the two, we arrive at a different conclusion. One such adjustment was done in the paper by Epstein and Zin (1989). They assume that preferences are recursively defined as

    $$U_t = \left[c_t^{1-\rho} + \beta \left(\mathbb{E}_t U_{t+1}^{1-\gamma} \right)^{\frac{1-\rho}{1-\gamma}} \right]^{\frac{1}{1-\rho}}$$

    Here ρ is the IES while γ can be thought of as risk-aversion. It turns out that again with these preferences, EP depends on risk-aversion and high values of it are required to match it.

2. Changes in the statistical processes:

    (a) Disaster: an idea dating back to Rietz (1988) and further developed by Barro (2006), assumes that there is always a small probability of a disaster or a tail event - a large downward shock that occurs rarely. Barro shows that by calibrating the probability of a tail event to observed disasters in the twentieth century, then the EPP is resolved.

    (b) Long-run risk: Bansal and Yaron (2004) assume that there are shocks to trend in dividends which make shocks to returns long lasting. Together with EZ preferences, they show that their model can explain the EPP.

3. Incomplete markets and heterogeneity: Another idea is that the assumption that the representative consumer is pricing these assets is not realistic. Perhaps, people who participate in the stock market are less risk-averse and they could potentially have a higher volatility in consumption. This idea has been explored in a paper by Guvenen (2009).

References

Abel, A. B. (2007): "Optimal capital income taxation," . 47

Alvarez, F. and N. L. Stokey (1998): "Dynamic Programming with Homogeneous Functions," *Journal of Economic Theory*, 82, 167–189. 23

Angeletos, G.-M. and J. La'O (2013): "Sentiments," *Econometrica*, 81, 739–779. 50

Atkinson, A. B. and J. E. Stiglitz (1976): "The design of tax structure: direct versus indirect taxation," *Journal of public Economics*, 6, 55–75. 42, 49

Bansal, R. and A. Yaron (2004): "Risks for the long run: A potential resolution of asset pricing puzzles," *The Journal of Finance*, 59, 1481–1509. 57

Barro, R. J. (2006): "Rare disasters and asset markets in the twentieth century," *The Quarterly Journal of Economics*, 823–866. 57

Barro, R. J. and G. S. Becker (1989): "Fertility Choice in a Model of Economic Growth," *Econometrica*, 57, 481–501. 25

Barro, R. J. and X. Sala-i Martin (1992): "Convergence," *Journal of Political Economy*, 223–251. 16

Blanchard, O. J. and C. M. Kahn (1980): "The Solution of Linear Difference Models under Rational Expectations," *Econometrica*, 48, 1305–1312. 54

Burns, A. F. and W. C. Mitchell (1946): "Measuring business cycles," *NBER Books*. 1

Campbell, J. Y. and J. H. Cochrane (1999): "By Force of Habit: A Consumption-Based Explanation of Aggregate Stock Market Behavior," *Journal of Political Economy*, 107. 57

Chamley, C. (1986): "Optimal Taxation of Capital Income in General Equilibrium with Infinite Lives," *Econometrica*, 54, 607–22. 48

Chari, V. and P. J. Kehoe (1990): "Sustainable Plans," *The Journal of Political Economy*, 98, 783–802. 48

Chatterjee, S. (1994): "Transitional dynamics and the distribution of wealth in a neoclassical growth model," *Journal of Public Economics*, 54, 97–119. 10

Deaton, A. (1979): "Optimally uniform commodity taxes," *Economics Letters*, 2, 357–361. 42, 46

——— (1981): "Optimal Taxes and the Structure of Preferences," *Econometrica*, 49, 1245–60. 42

Diamond, P. A. and J. A. Mirrlees (1971): "Optimal Taxation and Public Production I: Production Efficiency," *The American Economic Review*, 61, 8–27. 37, 42, 43, 44, 46

Dixit, A. K. and J. E. Stiglitz (1977): "Monopolistic Competition and Optimum Product Diversity," *American Economic Review*, 67, 297–308. 31

Duffie, D. and C.-F. Huang (1985): "Implementing Arrow-Debreu Equilibria by Continuous Trading of Few Long-Lived Securities," *Econometrica*, 53, 1337–1356. 51

Epstein, L. G. and S. E. Zin (1989): "Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework," *Econometrica*, 57, 937–969. 57

Grossman, G. M., E. Helpman, E. Oberfield, and T. Sampson (2016): "Balanced growth despite Uzawa," Tech. rep., National Bureau of Economic Research. 22

Guvenen, F. (2009): "A Parsimonious Macroeconomic Model for Asset Pricing," *Econometrica*, 77, 1711–1750. 58

Hall, R. E. and C. I. Jones (1999): "Why Do Some Countries Produce So Much More Output Per Worker Than Others?" *The Quarterly Journal of Economics*, 114, 83–116. 16

Hopenhayn, H. A. and E. C. Prescott (1992): "Stochastic Monotonicity and Stationary Distributions for Dynamic Economies," *Econometrica*, 60, 1387–1406. 53

Huo, Z. and N. Takayama (2015): "Higher Order Beliefs, Confidence, and Business Cycles," *Working Paper, Yale University*. 50

Jackson, M. O. and L. Yariv (2015): "Collective Dynamic Choice: The Necessity of Time Inconsistency," *American Economic Journal: Microeconomics*, 7, 150–178. 18

Judd, K. L. (1985): "Redistributive taxation in a simple perfect foresight model," *Journal of Public Economics*, 28, 59–83. 48

Karabarbounis, L. and B. Neiman (2014): "The Global Decline of the Labor Share*." *Quarterly Journal of Economics*, 129. 1, 15

Kushner, H. and P. G. Dupuis (2013): *Numerical Methods for Stochastic Control Problems in Continuous Time*, vol. 24, Springer Science and Business Media. 28

Laibson, D. (1997): "Golden eggs and hyperbolic discounting," *The Quarterly Journal of Economics*, 443–477. 10

Lucas Jr, R. E. (1978): "Asset Prices in an Exchange Economy," *Econometrica*, 46, 1429–1445. 55

Luenberger, D. G. (1969): *Optimization by Vector Space Methods*, John Wiley and Sons. 27

Mankiw, N. G., D. Romer, and D. N. Weil (1992): "A Contribution to the Empirics of Economic Growth," *The Quarterly Journal of Economics*, 107, 407–437. 23

Mehra, R. and E. C. Prescott (1985): "The Equity Premium: A Puzzle," *Journal of Monetary Economics*, 15, 145–161. 57

Piketty, T. (2014): "Capital in the Twenty-First Century," . 1

Ramsey, F. P. (1927): "A Contribution to the Theory of Taxation," *The Economic Journal*, 37, 47–61. 39

Rebelo, S. (1991): "Long-Run Policy Analysis and Long-Run Growth," *Journal of Political Economy*, 99, 500–521. 22

Rietz, T. A. (1988): "The Equity Risk Premium a Solution," *Journal of Monetary Economics*, 22, 117–131. 57

Romer, P. M. (1986): "Increasing Returns and Long-Run Growth," *The Journal of Political Economy*, 94, 1002–1037. 28

——— (1990): "Endogenous Technological Change," *Journal of Political Economy*, 98. 30

Santos, M. and M. Woodford (1997): "Rational Asset Pricing Bubbles," *Econometrica*, 65, 19–58. 54

Stokey, N. L. and E. Lucas Jr, Robert E with Prescott (1989): "Recursive Methods in Economic Dynamics," *Harvard University Press, Cambridge.* 12

Uzawa, H. (1961): "Neutral inventions and the stability of growth equilibrium," *The Review of Economic Studies*, 28, 117–124. 22